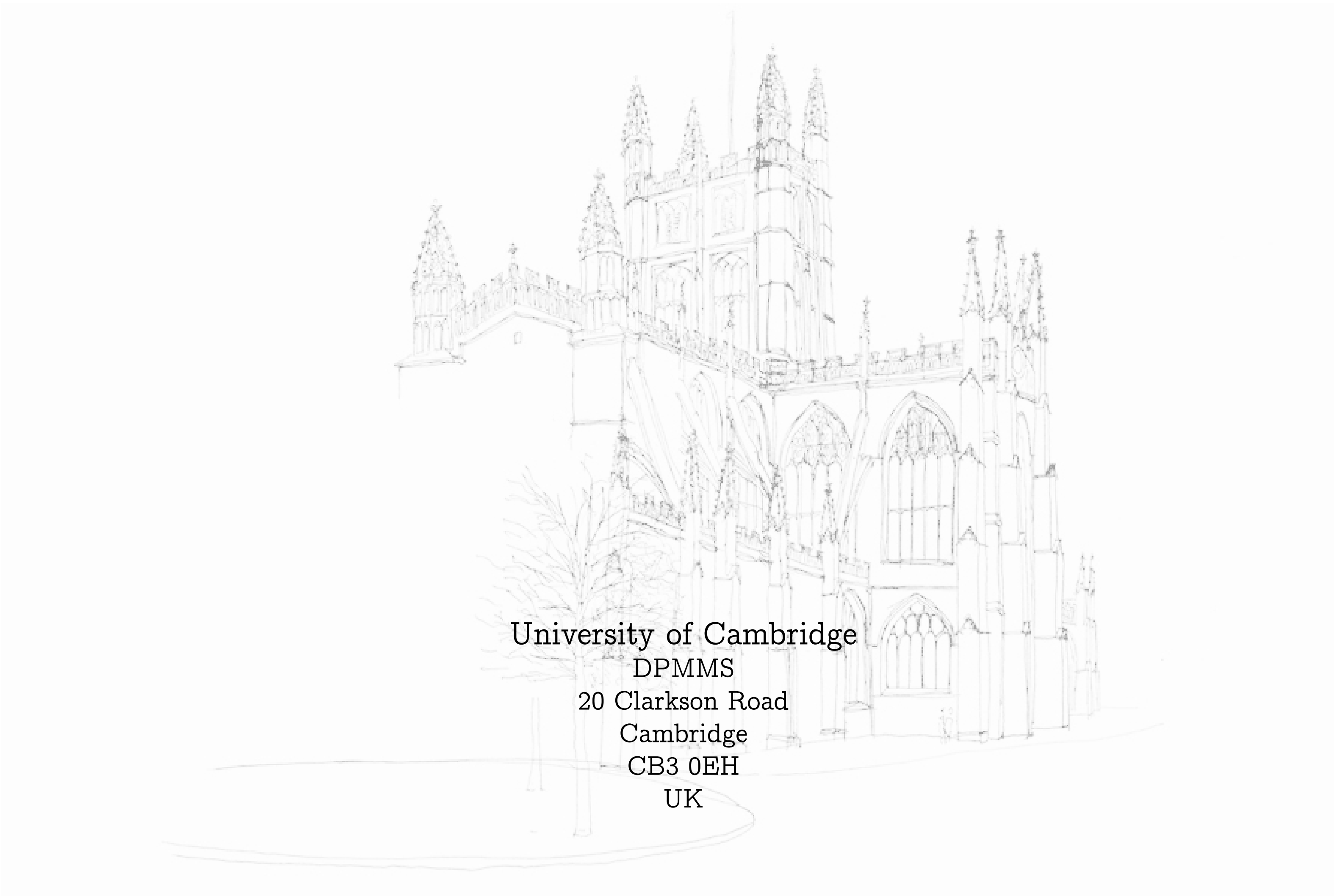
# Random Motions in Spacetime



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# Introduction

Probability theory has found many applications within analysis starting in the early $20^{\text{th}}$ century. The study of potential analysis using probability has resulted in more understanding and further development in that area ([Doo84], [Pin95]). Schramm-Loewner evolutions hallmarks a wonderful link between complex analysis and probability. There have also been probabilistic proofs of classic analytic results such as Liouville's theorem, Picard's little theorem ([Dav75]) and Atiyah-Singer index theorem ([Bis84]).

The natural question is then to ask if one can extend the use of probability to relativistic settings and deduce properties of the spaces that relativity is based on. The link between probability and analysis in the Euclidian setting stems from the one-to-one correspondence of bounded harmonic functions and random variables in the invariant $\sigma$-algebra,[1] which also holds in relativistic settings. Essentially, this is due to Brownian motion representing a particle of gas in an equilibrium system and harmonic functions represent the equilibrium state of the system. In relativistic settings, however, the classical means of specifying processes as having independent increments with a certain distribution fails as the notion of an increment no longer makes sense on manifolds. One is led to describing a diffusion through its properties analogous to $\mathbb{R}^d$, such as its generator and the invariance under the action of isometry groups.

The discourse into relativistic diffusions yields some strange results. For example, one can find a non-constant bounded harmonic function which is connected to the angular part of the diffusion converging. This lead to the study of the so called Poisson boundary, the set of non-constant bounded harmonic functions ([Bai08b], [BR08]). Indeed this has a strong connection to the asymptotic behaviour of relativistic diffusions via the link of the invariant algebra.

The paper investigates Markov processes in relativistic settings. The first account of this (as far as I am aware) is in [Dud65] where Dudley constructs and investigates the behaviour of Markov processes in Minkowski spacetime. The approach here will be different to that of Dudley, which was pointed out by my supervisor Bailleul. The construction of diffusions on a general Lorentz manifold is also shown, as this is a wonderful generalisation of the Minkowski case.

The first chapter gives a very brief account of geometry and probability tools that the reader may not be familiar with. I shall assume that the reader has a background to that

[1] See Chapter 4 for a definition.

comparable with the Part III courses advanced probability and stochastic calculus. The geometry knowledge required is minimum, and I have tried to give more or less intuitive explanations as opposed to heavy geometric arguments. Unfortunately the material in the first chapter alone could fill out a whole book, hence the reader is advised to acquaint themselves using a book on the subject where necessary. Chapter 2 will give an account of Lévy processes in Lie groups. These will be obtained as a solution to and SDE given in [AK93], and will also give a proof of the result in [Hun56] about the generators of Lévy processes in Lie groups. This will aid us in defining Markovian processes on relativistic settings as, for example, diffusions in Minkowski case can be thought as projection of diffusions from $SO^+(1, d)/SO(d)$. The main reference for this chapter is [Lia04], who in turn uses [AK93] and [Ram74] as his sources.

The remainder of the paper will be focused on relativistic diffusions, where in chapter 3 we give the construction of these on a Lorentz manifold and also some asymptotic properties of diffusions in Minkowski spacetime, namely that the diffusion asymptotically approaches a hyperplane at a random height. Recently [FLJ07] have also proved some properties of diffusions on a Schwarzschild spacetime, where the diffusion either hits the singularity in a finite time or tends off to infinity as $t \to \infty$ both with positive probability. The last chapter will give a informal account to the study of the Poisson boundary in the Minkowski setting.


## Acknowledgements

Undoubtedly the help of my supervisor Ismaël Bailleul was vital. I would like to give him many thanks for taking the time out to explain to me various concepts that are now collected in this essay. His enthusiasm and passion into the subject has inspired me greatly. Also I would like to thank my friends, and in particular those who are geometers and physicists for their endless patience under my bombardment of questions.

# PRELIMINARIES

| | |
|---|---|
| $\mathscr{C}(V)$ | the set of continuous functions from $V$ to $\mathbb{R}$ |
| $\mathscr{C}_b(V)$ | the set of bounded continuous functions from $V$ to $\mathbb{R}$ |
| $\mathscr{C}_c(V)$ | the set of continuous functions with compact support |
| $\mathscr{C}^k(V)$ | the set of k continuously differentiable functions from $V$ to $\mathbb{R}$ |
| $\mathscr{C}^\infty(V)$ | the set of smooth functions from $V$ to $\mathbb{R}$ |
| $T_pM$ | tangent space at $p \in M$ |
| $\circ\, dX$ | Stratonovich integral with respect to $X$ |
| $L_\tau$ | The left multiplication operator, i.e. $L_\tau(\sigma) = \tau\sigma$ |
| $\mathbb{R}_*$ | Strictly positive real numbers |

$TM := \coprod_{p \in M} T_pM$

## 1.1 Probability

A process $X$ on $(\Omega, \mathscr{F}, \mathbb{P})$ is said to be *Markovian* or a *Markov process* with respect to a filtration $\{\mathscr{F}_t\}_{t \geqslant 0}$ if the process is adapted to the filtration and for each $s, t \geqslant 0$ and $A \in \mathscr{F}$,

$$\mathbb{P}(X_{t+s} \in A | \mathscr{F}_s) = \mathbb{P}(X_{t+s} | \sigma(X_s)).$$

Recall that a process $X$ is a *strong Markov process* with respect to $\{\mathscr{F}_t\}_{t \geqslant 0}$ if for each stopping time $\tau$ with $\mathbb{P}(\tau < \infty) > 0$, we have that conditioned on $\{\tau < \infty\}$, the process $X_{\cdot+\tau} - X_\tau$ is independent of $\mathscr{F}_\tau$ and has the same law as $X_\cdot$. Here is a useful result from the study of Markov processes.

**Proposition 1.1.1.** *Any Markov process $X$ that is càdlàg is a strong Markov process.* [1]

*Proof.* Suppose that $\tau$ takes countably many values, then it is of the form $\sum t_n \mathbb{1}_{\tau = t_n}$. By conditioning on events such as $\{\tau = k\}$ and using the simple Markov property, we can deduce that $X_{\cdot+\tau}$ has the desired properties.

Now suppose that $\tau$ is any stopping time such that $\mathbb{P}(\tau < \infty) > 0$. We have that the result holds for each $\tau_n = 2^{-n}\lfloor 2^n\tau + 1\rfloor$ where $\lfloor x \rfloor$ is $x$ rounded down to the closest integer. As $\tau_n \downarrow \tau$, by using the right continuity we obtain the desired result. $\square$

Suppose we have a collection of measures $\{\mu_t\}_{t \geqslant 0}$ on a group $G$. We say that these satisfy the *semi-group* property if $\mu_t \star \mu_s = \mu_{t+s}$ where

$$\mu \star \nu(f) = \int f(\sigma\tau)\mu(d\sigma)\nu(d\tau).$$

[1] c.f. [Ber96]

A semi-group of measures if *weakly continuous* if $\lim_{t\downarrow 0}\mu_t = \delta_0$ in the weak sense. A weakly continuous semi-group of measures give rise to a *strongly continuous semi-group of operators* on $\mathscr{C}_b(G)$, that is a set of operators given by $P_t f(\sigma) = \mu_t(f \circ L_\sigma)$ for each $f \in \mathscr{C}_b(G)$, which have the property that $P_t P_s = P_{t+s}$ and $\lim_{t\downarrow 0} P_t f = f$ in the strong sense.[2]

Conversely each strongly continuous semi-group of operators have a unique weakly continuous semi-group of measures associated with them and if we assert that $||P_t|| = 1$ for each $t \geqslant 0$ then $\mu_t(G) = P_t(\mathbb{1}) = 1$, i.e. $\mu_t$ is a probability measure for each $t \geqslant 0$.

A *generator* of a semi-group of operators (or measures) is defined by;

$$\mathscr{L} f(\tau) = \lim_{t\downarrow 0} \frac{1}{t}[P_t f(\tau) - f(\tau)].$$

Essentially, the generator tells us about the infinitesimal action of the operator. We can also look at the generators of Markov processes. The processes that generate a strongly continuous semi-group on $\mathscr{C}_b(G)$ via $P_t f(\tau) = \mathbb{E}_\tau[f(X_t)]$ are called *Feller processes.* It is important to note that every Lévy process is a Feller process.[3]

Next is an important theorem which we will be needing.

**Theorem 1.1.2** (Hille-Yoshida). *If $\mathscr{L}$ is the generator of a weakly continuous semi-group of measures $\{\mu_t\}_{t\geqslant 0}$ on a group $G$, then $\mathscr{L}$ is closed and the domain of the generator is dense in $\mathscr{C}_b(G)$. Moreover two semi-groups of weakly continuous measures coincide if and only if their generators are the same.*

Also we give one proposition which follows directly from the uniqueness of generators.

**Proposition 1.1.3.** *If two strongly continuous semi-group of operators $\{P_t\}_{t\geqslant 0}$ and $\{Q_t\}_{t\geqslant 0}$ associated with the processes $X = (X_t : t \geqslant 0)$ and $Y = (Y_t : t \geqslant 0)$ are such that*

$$\lim_{t\downarrow 0} \frac{1}{t}[P_t f(x) - f(x)] = \lim_{t\downarrow 0} \frac{1}{t}[Q_t f(x) - f(x)] \qquad \forall x \in E, f \in \mathbb{D}$$

*then $X$ and $Y$ are modifications.*

The interested reader is referred to [Sat99] for a proof of both of these.

Stratonovich integrals are constructed in the same manner as the Itô integrals but instead for simple previsible processes $H$, we have

$$\int_0^t H_s \circ dX_s = \lim_{n\to\infty} \sum_{i=1}^{n-1} \frac{(H_{t_{i+1}} + H_{t_i})(X_{t_{i+1}} + X_{t_i})}{2}$$

[2] The strong sense refers to the fact that $\mathscr{C}_b(G)$ is a Banach space with the supremum norm and so the last statement reads that $||P_t f - f||_\infty \to 0$. This contrasts with the weak sense of convergence, which is that $\mu_n(f)$ converges to $\mu(f)$ for each $f \in \mathscr{C}_b(G)$.

[3] The Chapman-Kolmogorov equations give the semi-group property and the right continuity of the paths with the uniform continuity of $f$ gives the strong convergence using dominated convergence theorem, c.f. [Sat99].

and thus we can convert Stratonovich integral to an Itô integral using the following [4]

$$\int_0^t X_{s-} \circ dY_s = \int_0^t X_{s-} dY_s + \frac{1}{2}[X, Y]_t^c$$

where $X, Y$ are semi-martingales and $[X, Y]^c$ is the continuous part of the quadratic variation $[X, Y]$. Stratonovich equations are useful on manifolds as they provide a way of doing stochastic calculus without a dependency on the co-ordinates.[5]

## 1.2 Geometry

A $d$-manifold $M$ is a topological space that is locally homeomorphic to $\mathbb{R}^d$. That is, for each point $p \in M$, there is a neighbourhood $U$ of $p$ and a map $\pi : U \to \mathbb{R}^d$ such that $\pi(U)$ is open and $\pi$ is a homeomorphism to its image. Note that we can induce local co-ordinates on a manifold by looking at $\pi^{-1}(e_i)$ where $e_1, \cdots, e_d$ is the basis in $\mathbb{R}^d$. The system of the open sets along with the maps $\{(U, \pi)_\alpha\}_{\alpha \in I}$, such that $\{U_\alpha\}_{\alpha \in I}$ is a covering for $M$, is called an atlas. When we refer to a manifold we always assume that the atlas is given. The manifold is called smooth if there exists an atlas for which each pair of maps $\pi$ and $\pi'$ we have that the whenever the map $\pi' \circ \pi^{-1} : \pi(U \cap U') \to \mathbb{R}^d$ is well defined, it is smooth. We will always assume that our manifolds are connected, smooth and Hausdorff.

A manifold is Riemannian if it is equipped with a Riemannian metric $g$, which obeys the following;

(i) $g_p$ acts as an inner product on $T_pM \times T_pM$

(ii) For smooth vector fields $X, Y$, the map $p \mapsto g_p(X(p), Y(p))$ is smooth.

If the inner product is not positive definite then the metric is called pseudo-Riemannian and similarly, the space is called a pseudo-Riemannian manifold.

A connection on a manifold is used to connect the tangent spaces together. This is denoted by $\nabla$, and satisfies the following;

(i) $\nabla : \mathscr{C}^\infty(M, TM) \times \mathscr{C}^\infty(M, TM) \to \mathscr{C}^\infty(M, TM)$, $(X, Y) \mapsto \nabla_X Y$ is bilinear

(ii) For each $f \in \mathscr{C}^\infty(M, \mathbb{R})$ we have that $\nabla_{fX} Y = f \nabla_X Y$ and $\nabla_X fY = df(X)Y + \nabla_{fX} Y$.

The torsion of a connection is given by $\nabla_X Y + \nabla_Y X - [X, Y]$. On pseudo-Riemannian manifolds there exists a natural connection called the Levi-Civita connection, which is torsion free and preserves the inner product along parallel transports.

[4] Notice that the definition of the Stratonovich sum is nothing but the Itô sum with $H_{t_i}$ replaced with $(H_{t_{i+1}} + H_{t_i})/2$. [App04] serves as a good reference to all the statements made here.

[5] Note that the Itô integral may change (due to the quadratic variation terms) as the co-ordinate system of the space changes. This causes Itô integrals to be not well defined on manifolds.

An orthonormal frame is a pair $(p, s_p)$ such that $s_p = (s_p^1, ..., s_p^d)$ is an orthonormal basis for $T_pM$ with respect to $g_p$. The action of $O(n)$, the Euclidian rotations, is free and transitive on the orthonormal frame bundle.

A Lie group $G$ is a group equipped with the smallest smooth manifold structure such that the maps $(\sigma, \tau) \mapsto \sigma \cdot \tau$ and $\tau \mapsto \tau^{-1}$ are smooth. We will again be assuming that these are Hausdorff and connected. Each Lie group $G$ gives rise to a Lie algebra $\mathfrak{g}$ by taking the left invariant vector fields on it, and equipping it with the so called Lie bracket $[X, Y]$. The Lie algebra can also be identified with $T_eG$.

The group $O(1, d)$ is composed of $(d+1) \times (d+1)$ matrices $A$ with

$$A^T \begin{pmatrix} 1 & 0 & \cdots \\ 0 & -1 & \\ \vdots & & \ddots \end{pmatrix} A = \begin{pmatrix} 1 & 0 & \cdots \\ 0 & -1 & \\ \vdots & & \ddots \end{pmatrix}.$$

These can be though of rotations in the Lorentz manifold. The subgroup $SO(1, d)$ is then given by $\{A \in O(1, d) : det(A) = 1\}$ and has four connected components. The connected component with the identity is denoted by $SO^+(1, d)$.

# Chapter 2

# Lévy Processes in Lie Groups

Throughout this chapter we will take $G$ to be a locally compact, Hausdorff Lie group with Lie algebra $\mathfrak{g}$. We shall assume that $(\Omega, \mathscr{F}, \mathbb{P})$ is the probability space we are working in and that any filtration mentioned satisfies the "usual assumptions" of right continuity and completeness.[1]

We shall see how one can construct Lévy processes in groups. This will follow a mixture of [Lia04] and [AK93]. Lévy process are defined on a group using the increments specified by the group operator.

**Definition 2.0.1.** A *left Lévy process* (hereafter referred to as Lévy process) is a stochastic process $g = (g_t : t \geqslant 0)$ taking values in $G$ such that;

(i) $g_0 = \mathrm{e}$ a.s

(ii) The increments $g_{t_1}^{-1} g_{t_2}, \ldots, g_{t_{n-1}}^{-1} g_{t_n}$ are stationary and independent for each $0 \leqslant t_1 < t_2 < \cdots < t_n$.

(iii) The paths $t \mapsto g_t$ are càdlàg

The condition (iii) may be replaced with the weaker condition that the paths are continuous in probability (stochastically continuous),[2] in which case a càdlàg modification exists. Notice that by taking $G = (\mathbb{R}, +)$ we obtain the definition of a Lévy process on $\mathbb{R}$, so this definition seems reasonable.

The first section deals with Poisson processes in $G$ which we will need when we deal with the SDE given in the next section. The second section shows that a Lévy process can be given as a solution to an SDE. The last section proves the converse, that every Lévy process solves this SDE.

## 2.1 Poisson Processes

Suppose that $\{\xi_n : n \in \mathbb{N}\}$ is a set of i.i.d. $G$-valued random variables with law $F$ and that $K = (K_t : t \geqslant 0)$ is a Poisson process with parameter $\lambda > 0$. We define a *compound Poisson process* $Y$ via

$$Y_t = \xi_1 \cdot \xi_2 \ldots \xi_{K_t}$$

with the convention that the empty product (e.g. $Y_0$) is given the value $\mathrm{e}$. It is easily deducible that $Y$ is a Lévy process and moreover;

$$\mathbb{P}(Y_t \in da) = \sum_{n \geqslant 0} e^{-\lambda t} \frac{(\lambda t)^n}{n!} F^{*n}(da)$$

[1] Though in the case of Lévy processes, the natural filtration is right continuous, c.f. [Ber96]

[2] see [App04] for instance

where $F^{*n}$ is the $n$-fold convolution. We define a *Poisson random measure* on $G$ by the following construction;

$$N((0,t],A) = \#\{s \in [0,t) : \Delta Y_s \in A\}$$

where $\Delta Y_s = Y_s - Y_{s-}$, which extends by Kolmogorov's extension theorem to a measure on $\mathbb{R} \times G$. Via brute force computation, it can be shown that the number of jumps a $G$-valued Poisson process makes by time $t$ is Poisson with parameter $t\lambda$ and so the measure $N$ is indeed a Poisson random measure with intensity $Leb \otimes \Pi$ where $\Pi = \lambda F$.

**Lemma 2.1.1.** *The domain of the infinitesimal generator for $Y$ contains $\mathscr{C}_b(G)$ and moreover;*

$$\mathscr{L}f(a) = \int_{G\backslash\{e\}} [f(ab) - f(a)]\Pi(da) \tag{2.1.1}$$

*for each $f \in \mathscr{C}_b(G)$.*

*Proof.* Notice that for any $f \in \mathscr{C}_b(G)$ we have that

$$f(\sigma Y_t) = f(\sigma) + \sum_{0 \leqslant s \leqslant t} (f(\sigma Y_s) - f(\sigma Y_{s-})). \tag{2.1.2}$$

Now by noting the definition of the Poisson measure $N$ we see that the jumps of the process correspond to;

$$\sum_{0 \leqslant s \leqslant t} (f(\sigma Y_s) - f(\sigma Y_{s-})) = \int_0^t \int_{G\backslash\{e\}} [f(\sigma Y_s \tau) - f(\sigma Y_s)]\, N(ds, d\tau).$$

This is a martingale so has finite expectation, so by plugging it back into (2.1.2) and using Fubini's theorem we arrive at;

$$\mathbb{E}[f(\sigma Y_t)] - f(\sigma) = \int_0^t \int_{G\backslash\{e\}} (\mathbb{E}[f(\sigma Y_s \tau)] - \mathbb{E}[f(\sigma Y_s)])\, \Pi(d\tau) ds.$$

The result follows by differentiation. □

We shall be using the following proposition later which is an adaptation to that in [RY99].

**Proposition 2.1.2.** *Real valued Poisson processes $N_t^1, ..., N_t^n$ are independent if and only if they do not jump at the same time.*

*Proof.* To avoid heavy notation, we will prove this for $n = 2$ which easily extends to generality.

First let $N, N'$ be two independent real-valued Poisson processes and $T_n$ be the jump times of $N$. We have then that

$$\sum_{t \geqslant 0} (\Delta N_t)(\Delta N'_t) = \sum_{n \in \mathbb{N}} \Delta N'_{T_n}.$$

However $N'$ is independent from the stopping times $(T_n : n \in \mathbb{N})$ and also that for each fixed time $t$, $\Delta N_t = 0$. Hence we have that $\Delta N_{T_n} = 0$ for each $n \in \mathbb{N}$.

For the converse take Poisson processes $N, N'$ that do not jump together and two step functions $h, h'$. Consider the two exponential martingales given by

$$M_t = \exp\left(-\int_0^t h(s)dN_s + c\int_0^t (1 - e^{-h(s)})ds\right)$$
$$M_t = \exp\left(-\int_0^t h'(s)dN'_s + c'\int_0^t (1 - e^{-h'(s)})ds\right)$$

where $c, c'$ are constants.

Notice that $M$ and $M'$ never jump simultaneously either and are both are bounded with bounded variation. Using Ito's formula and the fact that $[N, N'] = \sum(\Delta N)(\Delta N') = 0$ we obtain that $MM' = (M_t M'_t : t \geqslant 0)$ is a martingale with $\mathbb{E}[M_t M'_t] = 1$. Thus

$$\begin{aligned}\mathbb{E}\left[\exp\left(-\int_0^t h(s)dN_s - \int_0^t h'(s)dN'_s\right)\right] &= \exp\left(c\int_0^t (1 - e^{-h(s)})ds\right)\exp\left(c'\int_0^t (1 - e^{-h'(s)})ds\right) \\ &= \mathbb{E}\left[\exp\left(-\int_0^t h(s)dN_s\right)\right]\mathbb{E}\left[\exp\left(-\int_0^t h'(s)dN'_s\right)\right]\end{aligned}$$

which shows the independence. $\square$

## 2.2 Stochastic Differential Equations

We will construct Lévy processes as a solution to some SDE. Indeed this is analogous to the case in $\mathbb{R}^d$, see for example [App04]. Firstly to define a "derivative" of some sort we must look to the Lie algebra. So we let $X_1, \ldots, X_d$ be a fixed basis of the Lie algebra for now. The basis induces local co-ordinates associated with them on the group, in the sense that $X_i x_j = \delta_{ij}$ via the following construction.

Set $F = \{f \in \mathscr{C}^2 : f(e) = X_i f(e) = 0 \,\text{for}\, i = 1, \ldots, d\}$ which is closed, non-empty and has finite co-dimension with respect to $\mathscr{C}^2$. Now we can explicitly construct a function $y \in \mathscr{C}^2$ such that $y(e) = 0$ and $X_i y(e) = 1$. Then we have that as $F + y$ is closed, $(F + y) \cap \mathbb{D}$ is dense in $\mathscr{C}^2$. Thus we have $x_1, \ldots, x_d \in \mathbb{D} \cap \mathscr{C}^2$ such that $x_i(e) = 0$ and $X_j x_i(e) = \delta_{ij}$ for each $i, j = 1, \ldots, d$.

The objective of this section is to obtain Lévy processes as a solution to the following SDE;

$$\begin{aligned}f(x_t) =& f(x_0) + \sum_{i=1}^d \int_0^t X_i f(x_{s-}) \circ dB_s^i + \sum_{i=1}^d c_i \int_0^t X_i f(x_{s-})\, ds \\ &+ \int_0^t \int_{G\backslash\{e\}} [f(x_{s-}\sigma) - f(x_{s-})]\, \overline{N}(ds, d\sigma) \\ &+ \int_0^t \int_{G\backslash\{e\}} \left[f(x_{s-}\sigma) - f(x_{s-}) - \sum_{i=1}^d x_i(\sigma) X_i f(x_{s-})\right] N(ds, d\sigma)\end{aligned} \tag{2.2.1}$$

where $f \in \mathscr{C}^2$.

For the remainder of the section we will assume that that $U$ is a relatively compact neighbourhood of e and denote by $\mathscr{F}_t$ the natural filtration with respect to $B$ and $N$. This next lemma allows one to connect the case on the group to that in $\mathbb{R}^d$. We leave the details of the Euclidian case out.

**Lemma 2.2.1.** *If supp$(\Pi) \subset U$ then SDE given by (2.2.1) has a unique local solution on $U$.*

*Proof.* Without a loss of generality we can assume that there exists a chart $(U, \psi)$ that maps the identity to the origin. Define $v_t = \psi(x_t)$, $N'(dt, A) = N(dt, \psi(A))$, $X_i'(\cdot) = D\psi[X_i\psi^{-1}(\cdot)]$, $X'(\cdot) = D\psi[\sum_{i=1}^d c_i X_i \psi^{-1}(\cdot)]$ and $v(x, y) = \psi(\psi^{-1}(x)\psi^{-1}(y))$.[3] The actions of these maps are smooth on $\mathbb{R}^d$, so

$$X_i' = \sum_{j=1}^d k_{ij} \frac{\partial}{\partial x_j} \quad X' = \sum_{j=1}^d b_j \frac{\partial}{\partial x_j}$$

where $k, b \in \mathscr{C}_c^\infty(\mathbb{R}^d)$ and $x_i$ are the local coordinates. Suppose we write out $f = (f_1, ..., f_d)$ and let $k_i = (k_{i1}, ..., k_{id})$ and $b = (b_1, ..., b_d)$, then (2.2.1) becomes;

$$\begin{aligned} y_t =& \sum_{i=1}^d \int_0^t k_i(y_{s-}) \circ dB_s^i + \int_0^t b(y_s) ds \\ &+ \int_0^t \int_{\psi(U)\setminus\{0\}} [v(y_{s-}, x) - y_{s-}] \, \overline{N}'(ds, dx) \\ &+ \int_0^t \int_{\psi(U)\setminus\{0\}} [v(y_{s-}, x) - y_{s-} - \sum_{i=1}^d x_i k_i(y_{s-})] \, N'(ds, dx). \end{aligned} \tag{2.2.2}$$

This is a well known SDE that has a unique solution that is adapted to the filtration of $B$ and $N'$. See [App04][chap. 6] for details. As $\psi$ is bijective, the uniqueness carries forward to $G$.

□

It follows from Kolmogorov's theorem that we can specify any process with the topology of the space and so the solution to (2.2.2) when mapped back to $G$ will be adapted to $\mathscr{F}_t$ as $\psi$ is homeomorphic.

The following lemma will allow us to patch up the solutions to obtain a global solution.

**Lemma 2.2.2.** *Suppose that $g_t$ is a solution of (2.2.1) for some $B$ and $N$ and let $N' = Leb \otimes \Pi'$, where $\Pi'(G) < \infty$ and it is independent of $B$ and $N$. Let $J_n$ be the $n$-th jump time and $a_n$ the corresponding jump size of $N$.*

[3] $D$ is the derivative operator.

*Define $y$ inductively by*

$$\begin{aligned} y_t &= g_t && \text{if } 0 \leqslant t < J_1 \\ y_t &= y(J_n-)a_n g(J_n)^{-1} g_t && \text{if } J_n \leqslant t < J_{n+1}. \end{aligned}$$

*Then $y_t$ is a solution to (2.2.1) with $N$ and $c_i$ replaced by $N + N'$ and $c_i + \int x_i d\Pi'$.*

*Proof.* First, note that $y_0 = 0$ and that $t \mapsto y_t$ is càdlàg for $J_n \leqslant t < J_{n+1}$. By the right-continuity of $g_t$ we have that $y(J_n+) = y(J_n-)a_n g(J_n)^{-1} g(J_n) = y(J_n-)a_n = y(J_n)$ and also a similar calculation shows that the left limits are finite, thus $t \mapsto y_t$ is càdlàg. As $N$ is a Poisson measure, the jump times and sizes are independent and stationary, and noting that $g_t$ also has independent stationary increments leads to the conclusion that $y = (y_t : t \geqslant 0)$ is a Lévy process.

Suppose we write

$$\begin{aligned} I(f, g, a, b) = &\sum_{i=1}^{d} \int_a^b X_i f(g_{s-}) \circ dB_s^i + \sum_{i=1}^{d} c_i \int_a^b X_i f(g_{s-})\, ds \\ &+ \int_a^b \int_{G\backslash\{e\}} [f(g_{s-}\sigma) - f(g_{s-})]\, \overline{N}(ds, d\sigma) \\ &+ \int_a^b \int_{G\backslash\{e\}} \left[ f(g_{s-}\sigma) - f(g_{s-}) - \sum_{i=1}^{d} x_i(\sigma) X_i f(g_{s-}) \right] N(ds, d\sigma). \end{aligned}$$

for $a < b$ and $f \in \mathscr{C}^2$.

Now as the jumps of $g$ are determined by $N$ and this is independent from $N'$. Thus $g$ is left continuous at $t = J_1$ and hence continuous there and so we have that for $J_1 \leqslant t < J_2$,

$$f(y_t) = f(g(J_1)) + [f(g(J_1)a_1 g(J_1)^{-1} g_t) - f(g(J_1)a_n)] + [f(g(J_1)a_1) - f(g(J_1))]. \quad (2.2.3)$$

Note that by definition we have

$$f(g(J_1)a_1) - f(g(J_1)) = \int_0^t \int_{G\backslash\{e\}} [f(y_{s-}\sigma) - f(y_{s-})] N'(ds, d\sigma).$$

Also

$$\begin{aligned} f(g(J_1)a_1 g(J_1)^{-1} g_t) - f(g(J_1)a_n) &= (f \circ L_{g(J_1)a_1 g(J_1)^{-1}})(g_t) - (f \circ L_{g(J_1)a_1 g(J_1)^{-1}})(g_{J_1}) \\ &= I(f \circ L_{g(J_1)a_1 g(J_1)^{-1}}, g, J_1, t) = I(f, y, J_1, t) \end{aligned}$$

where the last equality follows from the definition of $I$.

By the strong Markov property, the processes $B^{J_1} := B - B_{J_1}$ and $N^{J_1}([0,t], \cdot) := N([J_1, J_1 + t], \cdot)$ are distributed as $B$, $N$ respectively and moreover they are independent of $\mathscr{F}_{J_1}$. Hence by using this and plugging in the above two calculations to (2.2.3) we have for $J_1 \leqslant t < J_2$;

$$f(y_t) = f(g(J_1)) + I(f, y, J_1, t) + \int_0^t \int_{G\backslash\{e\}} [f(y_{s-}\sigma) - f(y_{s-})] N'(ds, d\sigma).$$

But then as $f(g(J_1)) = f(g_0) + I(f, g, 0, J_1) = f(y_0) + I(f, y, 0, J_1)$, we have that

$$f(y_t) = f(y_0) + I(f, y, 0, t) + \int_0^t \int_{G\setminus\{e\}} [f(y_{s-}\sigma) - f(y_{s-})] N'(ds, d\sigma)$$

which shows the result for $J_1 \leqslant t < J_2$. The general result is obtained by carrying on inductively. □

The theorem now follows immediately.

**Theorem 2.2.3.** *The SDE (2.2.1) has a unique solution which is a Lévy process on $G$ started from $g_0$.*

*Proof.* From above we know that a solution exists in a relatively compact neighbourhood $U$ around the identity. Then by letting $N = N|_U$ and $N' = N|_{U^c}$, the second lemma gives the existence of a global solution. □

## 2.3 Generators

The remainder of this chapter will go towards proving the following theorem.

**Theorem 2.3.1.** *For every Lévy process, there exists a modification that solves (2.2.1).*

With the proof of the theorem, we will also prove the expression of the generators of Lévy processes in Lie groups which was first proved by [Hun56].

First let us denote the generator;

$$\mathscr{L} f(x) := \lim_{t\downarrow 0} [P_t f(x) - f(x)]$$

where $P_t f(x) = \mathbb{E}_x[f(g_t)]$.

Now define $\mathbb{D}$ to be the domain of $\mathscr{L}$. Then we have the following lemma.

**Lemma 2.3.2.** *$\mathbb{D} \cap \mathscr{C}^2$ is dense in $\mathscr{C}^2$.*

*Proof.* Suppose that $f \in \mathscr{C}^2$ and $X, Y \in \mathfrak{g}$, then notice that $XYP_t f = P_t XYf$ and so $P_t f \in \mathscr{C}^2$ for each $t \geqslant 0$. As a direct consequence we see that $||P_t f||_2 \leqslant ||f||_2$ and so by an application of the Hille-Yoshida theorem the result follows. □

There is a generalisation from the preceding lemma that we shall make, that is to assume that the generator is well defined for all $f \in \mathscr{C}^2(G)$. The reader is directed to [Lia04] for a verification.

In order to prove that a Lévy process solves the SDE, we need $a_{ij}$ to specify a Brownian motion, $c_i$ to specify the drift and a Poisson measure $N$. This is easy if we consider the Lévy-Itô decomposition, which states that a Lévy process is a Brownian motion with drift plus a pure jump process. We do not, however, have access to this theorem yet, but it shall do no harm to bear that in mind in what follows.

To that end, define

$$N((0,t],B) = \#\{s \in (0,t] : g_{s-}^{-1} g_s \in B \backslash \{\mathrm{e}\}\}$$

which count the number of jumps that land in $B$.

**Lemma 2.3.3.** *The measure $N$ is a Poisson random measure, and moreover the intensity of the jumps are given by $\Pi$ satisfying the following;*

(i) $\Pi(\mathrm{e}) = 0$

(ii) $\Pi(\mathbb{1}_K \sum_{i=1}^d x_i^2) < \infty$ *for any compact $K$*

(iii) $\Pi(U^c) < \infty$ *for any neighborhood $U$ of* $\mathrm{e}$.

(iv) $\Pi(\phi) = \mathscr{L}\phi(\mathrm{e})$ *for any $\phi \in \mathscr{C}_c(G)$ with $\phi(\mathrm{e}) = 0$.*

*Proof.* Without loss of generality we may assume that $g_0 = \mathrm{e}$ almost surely. First, for any $B \in \mathscr{F}$ fixed, $t \mapsto N_t^B := N((0,t],B)$ is càdlàg as $t \mapsto g_t$ is and also that the increments are independent and identical. The process increments by 1, and so each point is a holding point. As the increments are identical and independent, the holding points must be memoryless and thus $(N_t^B : t > 0)$ is a Poisson process for each $B \in \mathscr{F}$.

It is clear that for disjoint sets $B_1, ..., B_n$, $N_t^{B_1}, ..., N_t^{B_n}$ cannot jump together (as that would imply that $g$ jumps into disjoint sets at the same time, which is absurd). Thus by Proposition 2.1.2 $N_t^{B_1}, ..., N_t^{B_n}$ are independent, and so $N$ is a Poisson measure. Let $\Pi$ be its intensity. We will now show the properties of $\Pi$ as claimed.

Notice that (i) is immediate and that (ii) and (iii) follow easily from (iv), so we shall only prove (iv). It suffices to show that (iv) holds for any $\phi \in \mathscr{C}_c^\infty(G)$ that vanishes on a neighbourhood $U$ of $\mathrm{e}$ and $0 \leqslant \phi \leqslant 1$. Recall that the intensity measure can be computed by $\Pi(\phi) = \mathbb{E} \int_0^1 \int_G \phi(\sigma) N(dt, d\sigma)$.

Notice that $\phi(g_s^{-1} g_t) = \int_s^t \int_G \phi(\sigma) N(du, d\sigma)$ and so we have that

$$f_n := \sum_{k=1}^n \phi(g_{(k-1)/n)}^{-1} g_{k/n}) \to \int_0^1 \int_G \phi(\sigma) N(dt, d\sigma) \quad \text{a.s.}$$

Thus by stationary and independent increments;

$$\begin{aligned}
\mathbb{E}[f_n^2] &= \sum_{j,k=1}^n [\phi(g_{(k-1)/n)}^{-1} g_{k/n}) \phi(g_{(j-1)/n)}^{-1} g_{j/n})] \\
&= \sum_{j \neq k} \mathbb{E}[\phi(g_{(k-1)/n)}^{-1} g_{k/n})] \mathbb{E}[\phi(g_{(j-1)/n)}^{-1} g_{j/n})] + \sum_{l=1}^n \mathbb{E}[\phi(g_{(l-1)/n)}^{-1} g_{l/n})^2] \\
&= \sum_{j \neq k} \mathbb{E}[\phi(g_{1/n)})]^2 + \sum_{l=1}^n \mathbb{E}[\phi(g_{1/n})^2] = (n-1)n\mathbb{E}[\phi(g_{1/n)})]^2 + n\mathbb{E}[\phi(g_{1/n)})^2].
\end{aligned}$$

Noting that $\phi(e) = 0$ gives that $(n-1)n\mathbb{E}[\phi(g_{1/n})] = (n-1)nP_{1/n}\phi(\mathrm{e}) \to \mathscr{L}\phi(\mathrm{e})$ a.s. as $n \to \infty$, and similarly for the second term. So

$$\mathbb{E}[f_n^2] \to (\mathscr{L}\phi(\mathrm{e}))^2 + \mathscr{L}\phi^2(\mathrm{e}) < \infty$$

and so the family $\{f_n : n \geqslant 1\}$ is bounded in $L^2$, and hence, uniformly integrable. This implies that $f_n \to \int_0^1 \int_G \phi(\sigma)N(dt, d\sigma)$ in $L^1$ and so we have just proved the following;

$$\Pi(\phi) = \lim_{n\to\infty} \mathbb{E}[f_n] = \mathscr{L}\phi(\mathrm{e}).$$

□

*Remark* 2.3.4. The measure $\Pi$ above is called the *Lévy measure* of $g$. It can also be obtained by the fact that $f \mapsto \mathscr{L}f(\mathrm{e})$ is a linear map which is bounded, and so the Riesz representation theorem, there exists a unique Borel measure $\overline{\Pi}$ on the one point compactification $G_c$ of $G$ such that $\overline{\Pi}(\{\infty\}) = \overline{\Pi}(\{\mathrm{e}\}) = 0$ and $\Pi = \overline{\Pi}|_G$ gives us the measure we have described above, c.f. [Lia04].

**Lemma 2.3.5.** *Let $\{x_i\}_{i=1}^d$ be the co-ordinates as before and $g = (g_t : t \geqslant 0)$ be a Lévy process, then*

$$\sup_{t>0} \frac{1}{t}\mathbb{E}\left[\sum_{i=1}^d x_i(g_t)^2\right] < \infty.$$

*Proof.* By 2.3.3 we have that $\lim_{t\downarrow 0}(1/t)\mathbb{P}(g_t \in U^c) < \infty$ for every neighbourhood $U$ around the identity and for each $t > 0$, $(1/t)\mathbb{P}(g_t \in U^c) < \infty$. Then

$$\sup_{t>0} \frac{1}{t} \int \mathbb{1}_{U^c}(\sigma)\mathbb{P}(g_t \in d\sigma) < \infty.$$

Similarly taking $U$ compact gives that $\sup_{t>0} \frac{1}{t}\mathbb{E}[\mathbb{1}_U \sum x_i(g_t)^2] < \infty$ and the result follows. □

Let $a_{ij} = \mathscr{L}(x_i x_j)(\mathrm{e}) - \Pi(x_i x_j)$, $c_i = \mathscr{L}(x_i)(\mathrm{e})$ where $\Pi$ is as defined in Lemma 2.3.3. Clearly the matrix $(a_{ij})_{1\leqslant i,j\leqslant d}$ is symmetric and also for any $k \in \mathbb{R}^d$ we have

$$\mathscr{L}\left(\sum_{i,j=1}^d k_i x_i k_j x_j\right)(\mathrm{e}) \leqslant \mathscr{L}\phi_n\left(\sum_{i=1}^d k_i x_i\right)^2(\mathrm{e})$$

where $\phi_n \in \mathscr{C}_c^\infty$ with $0 \leqslant \phi_n \leqslant 1$ and $\{\sigma : \phi_n(\sigma) \neq 0\} \uparrow G$. By tending to the limit, the right hand side tends to $\Pi(x_i x_j)$, which gives that $(a_{ij})$ is positive definite.

Thus we can construct a Brownian motion $B$ with using $(a_{ij})_{1\leqslant i,j\leqslant d}$ as the covariance matrix.

*Proof of Theorem 2.3.1.* Using the notation above we will first prove that the generator of $g_t$ is uniquely determined by $a_{ij}$, $c_i$ and $\Pi$. It suffices to consider the generator evaluated at $\mathrm{e}$, as we can substitute $f = f \circ L_\tau$ to get its value at $\tau$. First consider the Taylor series of $f(\sigma \exp(s \sum x_i X_i))$ which gives;

$$f(\sigma\tau) - f(\sigma) = \sum_{i=1}^{d} x_i(\tau) X_i f(\sigma) + \frac{1}{2} \sum_{i,j=1}^{d} x_i(\tau) x_j(\tau) X_i X_j f(\tau\sigma')$$

where $\sigma' = \exp(s \sum x_i(\sigma) X_i)$ for some $s \in [0,1]$. By plugging in $\tau = g_t$ and recalling that $x_i(\mathrm{e}) = 0$, we have

$$\mathbb{E}[f(\sigma g_t) - f(\sigma)] = \sum_{i=1}^{d} \mathbb{E}[x_i(g_t) - x_i(e)] X_i f(\sigma) + \frac{1}{2} \sum_{i,j=1}^{d} \mathbb{E}[x_i(g_t) x_j(g_t) X_i X_j f(\tau g_t')].$$

We needn't worry about the term $\sum_{i=1}^{d} \mathbb{E}[x_i(g_t) - x_i(e)] X_i f(\sigma)$ as it is clear that by dividing by $t$ and tending to the limit, we see that this is determined by $\{c_i\}_{i=1}^{d}$.

For the last term we can separate the sum as

$$\sum_{i,j=1}^{d} x_i(g_t) x_j(g_t) X_i X_j f(\tau g_t') = \sum_{i,j=1}^{d} x_i(g_t) x_j(g_t) X_i X_j f(\mathrm{e}) + \sum_{i,j=1}^{d} x_i(g_t) x_j(g_t) X_i X_j (f(g_t') - f(\mathrm{e})).$$

By taking expectations and limits gives

$$\begin{aligned} \mathscr{L}\left[\sum_{i,j=1}^{d} x_i x_j X_i X_j f(\tau)\right](\mathrm{e}) &= \sum_{i,j=1}^{d} a_{ij} X_i X_j f(\mathrm{e}) + \sum_{i,j=1} \Pi(x_i x_j) X_i X_j f(\mathrm{e}) \\ &\quad + \mathscr{L}\left[\sum_{i,j=1}^{d} x_i x_j X_i X_j (\overline{f} - f(\mathrm{e}))\right](\mathrm{e}). \end{aligned}$$

where $\overline{f}(\tau) = f(\tau')$.

Notice that $A(\sigma) := x_i(\sigma) x_j(\sigma) X_i X_j (\overline{f}(\sigma) - f(\mathrm{e})) \in \mathscr{C}(G)$ and $A(\mathrm{e}) = x_i(\mathrm{e}) x_j(\mathrm{e}) X_i X_j (f(\mathrm{e}) - f(\mathrm{e})) = 0$. As the local co-ordinate functions $x_i$ determine a co-ordinate system around $\mathrm{e}$, there exists a relatively compact neighbourhood $W$ around the identity such that with $\phi_n$ as above, we have that

$$\mathscr{L}(A - \phi_n A)(\mathrm{e}) \leqslant \sup_{t>0} \frac{1}{t} \mathbb{E}[(1 - \phi(g_t)) |A(g_t)|] \sim \sup_{t>0} \frac{1}{t} \mathbb{E}\left[(1 - \phi_n(g_t)) \sum_{i=1}^{d} x_i(g_t)^2\right].$$

Now as $\{\phi_n < 1\} \downarrow \{\mathrm{e}\}$ and $x_i(\mathrm{e}) = A(\mathrm{e}) = 0$, from Lemma 2.3.5 it follows that $\mathscr{L}(A - \phi_n A) \to 0$ as $n \to \infty$.

Hence we can conclude that the generator of $g_t$ is uniquely determined (up to modification) by $(a_{ij})$, $c_i$ and $\Pi$.

From Theorem 2.2.3 we can deduce that there exists a solution to (2.2.1) with $B$ (having $(a_{ij})$ as its covariance matrix), $c_i$ and $N$ (Poisson measure with intensity $\Pi$) given above. Noting that the compensated Poisson process $\overline{N}$ has constant expectation and that in local co-ordinates the Brownian has generator $\sum_{ij} a_{ij}X_iX_jf$ gives the following expression of the generator for the solution to the SDE;

$$\begin{aligned}\mathscr{L}f(\tau) &= \sum_{i=1}^{d} c_iX_if(\tau) + \sum_{i,j=1}^{d} a_{ij}X_iX_jf(\tau) \\ &\quad + \int_{G\setminus\{e\}} (f(\tau\sigma) - f(\tau) - \sum_{i=1}^{d} X_if(\tau)x_i(\sigma))\,\Pi(d\sigma).\end{aligned}$$

So then the generator of the solution is completely determined by $a_{ij}$, $c_i$ and $\Pi$. The result now follows from the uniqueness of generators, Prop. 1.1.3. □

The preceding theorem and Theorem 2.2.3 give the following result which was first established in [Hun56].

**Corollary 2.3.6** (Hunt). *A right continuous process $g = (g_t : t \geqslant 0)$ is a Lévy process if and only if it has a generator of the form*

$$\begin{aligned}\mathscr{L}f(\tau) &= \sum_{i=1}^{d} c_iX_if(\tau) + \sum_{i,j=1}^{d} a_{ij}X_iX_jf(\tau) \\ &\quad + \int_{G\setminus\{e\}} (f(\tau\sigma) - f(\tau) - \sum_{i=1}^{d} X_if(\tau)x_i(\sigma))\,\Pi(d\sigma).\end{aligned}$$

*where $(a_{ij})$ is symmetric and positive definite and $\Pi$ is a Lévy measure.*

*Remark* 2.3.7. So far the dependence of the co-ordinates have been left out. Notice that if $\{x_i'\}$ are new co-ordinate for the same basis $X_1, \dots, X_d$ then $x_i' = x_i + o(x^2)$, moreover if $a_{ij}' = \mathscr{L}(x_i'x_j')(\mathrm{e}) - \Pi(x_i'x_j')$ then

$$\mathscr{L}(x_i'x_j')(\mathrm{e}) = a_{ij}' + \Pi(x_i'x_j') = a_{ij}\Pi(x_i'x_j')$$

and so $a_{ij} = a_{ij}'$.

Indeed it turns out that the diffusion component $\sum_{i,j=1}^{d} a_{ij}X_iX_j$ is also invariant under basis changes (c.f. [Lia04][Prop. 1.3]).

*Remark* 2.3.8. We will from now on be assuming that every Lévy process solves the SDE. This is not really a hindrance as we will be considering the so called Lévy triplets $(a, b, \Pi)$ which determine the Lévy process. To the reader that is uncomfortable with this notion, formally we can create equivalence classes of Lévy processes with respect to modifications and choose the representatives that solve the SDE.

# Stochastic Processes in Spacetime

The study of Markov process in spacetime invariant under isometries can be found in [Dud65]. His original construction is different to that given in this chapter. The first section describes two methods of construction of diffusions in Minkowski spacetime. The second method will be extended to the general case of curved spacetime, using more tools from geometry. The last two sections describe the asymptotic behaviour of diffusions in the Minkowski spacetime. The sections 1,3,4 use [BR08], [Bai08b] and [Bai08a] as their main reference. The second section derives from [FLJ07] and [Hsu02].

In favour of less notation we leave out the possible explosion of the processes described in this chapter, though it is worth to note that in the Minkowski setting diffusions do have infinite lifetime.

## 3.1 Minkowski Spacetime

In the theory of relativity, one includes time as a part of space on the same manifold. The Minkowski spacetime is a flat manifold, denoted by $\mathbb{R}^{1,d}$ which is indeed as a set the same as $\mathbb{R}^{1+d}$, however, the first component denotes the "time" of the object and the pseudo-Riemannian metric is given by $q(x) = x_0^2 - \sum_{i=1}^{d} x_i^2$ for $x \in \mathbb{R}^{1,d}$. The theory of relativity asserts that only massless objects may travel at the speed of light and nothing can travel faster than the speed of light. We shall model diffusions of objects that have mass and thus must travel strictly less than the speed of light. We adhere to common convention in relativity that the speed of light $c = 1$.

Generally on manifolds there is no notion of an increment and hence defining stochastic processes cannot be done by classical means. There is, however, a generalised notion of a Laplacian on any pseudo-Riemannian manifold called the Laplace-Beltrami operator, which is given by $\Delta_M f := div\, grad\, f$. As in the Euclidian case, the diffusions are associated with generators of the form $(\sigma^2/2)\Delta_M$. Feller process in Euclidian settings naturally give rise to Lévy process as any càdlàg Feller process invariant under the isometries is a Lévy process (c.f. [Sat99]). Thus it would seem natural then to try and find càdlàg Feller processes on $\mathbb{R}^{1,d}$.

The direct isometry group of $\mathbb{R}^{1,d}$ is $SO(1,d)$ which has four connected components. $SO^+(1,d)$ denotes the identity component in $SO(1,d)$ and consists of isometries that don't change the direction of time nor change the orientation of space.

There is a problem that is encountered in relativistic settings when we ask for the process to have speed less than that of light. For example, an object such as Brownian motion is nowhere differentiable and the small time boosts behave quite frantically, and so it would be unrealistic to expect that it indeed conforms to the principles of relativity. In [Dud65] Dudley proves that it is impossible to have a non-trivial càdàg Feller process invariant under $SO^+(1,d)$. It is essentially due to the fact that in order to assert conditions on the velocity of a process at time $t$, one has to assert conditions on $\mathscr{F}_{t-}$ which means that the process is no longer Markovian unless its filtration is left continuous. The solution to this problem is to consider a càdlàg process $\dot{\xi}_s$ on $\{x \in T\mathbb{R}^{1,d}; q(x) < 1, x_0 > 0\}$ and then let $\xi_t = \xi_0 + \int_0^t \dot{\xi}_{s-} ds$ to obtain a process on $\mathbb{R}^{1,d}$.

In relativity theory, each frame is relative to the object that is in spacetime. Each object carries its own "time clock" which leads to counter-intuitive results such as the famed twin paradox. It is often helpful to parametrise the process by the right inverse of $t \mapsto \int_0^t \sqrt{(\dot{\xi}_s)} ds$ which gives a process on the hyperbolic space $\mathbb{H} := \{x \in \mathbb{R}^{1,d} : q(x) = 1, x_0 > 0\}$. The correspondence here is one-to-one so henceforth we shall be assuming that $(\xi_s, \dot{\xi}_s) \in \mathbb{R}^{1,d} \times \mathbb{H}$ with $\xi_t = \xi_0 + \int_0^t \dot{\xi}_{s-} ds$.

If we fix the point $(1, 0, 0, 0)$ as the origin and look at the action of $SO^+(1, d)$ on $\mathbb{H}$, two things become apparent. First is that the action of $SO^+(1, d)$ is transitive and free on $\mathbb{H}$, and secondly that the origin has stabilizer $SO(d)$.[1] This allows for the identification of $\mathbb{H} \simeq SO^+(1, d)/SO(d)$. The action of $SO^+(1, d)$ on $\mathbb{H}$ can be viewed as the left action on the cosets $gSO(d)$, and this establishes a relation between process on the Lie group $SO^+(1, d)$ and processes on $\mathbb{H}$. The next theorem describes how to obtain Markov processes in $\mathbb{H}$ via the action of $SO^+(1, d)$. We shall use a more geometric approach to construct diffusions, so the proof is left out.

**Theorem 3.1.1** ([Lia04]Theorem 2.2 p.43). *Let $\tau : SO^+(1, d) \to \mathbb{H}$, $h = (h^0, \dots, h^d) \mapsto h^0$. If $g = (g_t : t \geqslant 0)$ is an $SO(d)$-invariant Lévy process on $SO^+(1, d)$ then $\tau(g_t)$ is a càdlàg $SO^+(1, d)$-invariant Feller process on $\mathbb{H}$.*

*Conversely, for any càdlàg $SO^+(1, d)$-invariant Feller process with laws $\{\mu_t\}_{t \geqslant 0}$ on $\mathbb{H}$ there exists an $SO(d)$-invariant Lévy process $g = (g_t : t \geqslant 0)$ on $SO^+(1, d)$ such that for each $t \geqslant 0$ the measure $\mu_t$ is the image measure of the law of $g_t$ under $\tau$.*[2]

There is an alternative construction which extends to general Lorentz manifolds in the next section. The Poincaré group $\mathcal{P}$ is the group of affine isometries of $\mathbb{R}^{1,d}$ and it is identified with $SO^+(1, d) \times \mathbb{R}^{1,d}$ as it is the Lorentz group with a shift. Let $\mathbb{O}M$ be the orthonormal frame bundle with the first element in $\mathbb{H}$, then it can be identified with the Poincaré group $\mathcal{P}$ where $\mathbb{R}^{1,d}$ represents the points and $SO^+(1, d)$ represents the frames.[3]

Let $e_0, \dots, e_d$ be the canonical basis for $\mathbb{R}^{1,d}$ and $e_0^*, \dots, e_d^*$ be the corresponding dual basis with respect to $q$, viz. $e_i = (0, \dots, 1, 0, \dots)^T$ and $e_0^* = e_0^T$ and $e_i^* = (0, \dots, -1, 0 \dots)$. Define $E_{ij} := e_i^* \otimes e_j - e_j^* \otimes e_i$ for $i, j = 0, \dots, d$. These matrices generate $so^+(1, d)$, the Lie algebra of $SO^+(1, d)$, where $E_i := E_{0i}$ correspond to the boosts transformations, i.e. the Lie algebra of $SO^+(1, d)/SO(d)$. Then from the previous chapter, a Brownian motion

[1] A stabilizer is the subgroup that leaves the point invariant. This always allows for the identification given. In this case, it is easy to see that the origin is fixed by Euclidian rotations.

[2] The map $\tau$ can also be thought of as the left action map on the left cosets of $SO^+(1, d)/SO(d)$.

[3] The frames are identified with $SO^+(1, d)$ in the following way: suppose $s^0, \dots, s^d$ is an orthonormal frame at a point $p$, and let $e_0, \dots, e_d$ be the canonical basis for $\mathbb{R}^{1,d}$ (which is the same as the canonical basis for $\mathbb{R}^{d+1}$). Because the action of $SO^+(1, d)$ on $\mathbb{R}^{1,d}$ is transitive, we can find an element of $SO^+(1, d)$ that maps the canonical basis to the given orthonormal one.

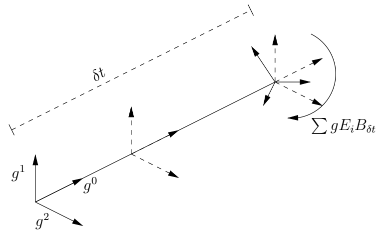


Figure 3.1: Diffusion on the frame bundle.

$(g_s, \zeta_s) \in \mathcal{P}$ solves

$$\begin{aligned} dg_s &= \sum_{i,j=1}^{d} g_s^i E_{ij} \circ dB_s^{i,j} \\ d\zeta_s &= g_s^0 ds \end{aligned}$$

as the matrices $E_{ij}$ together with $e_0$ form a basis for the Lie algebra of $\mathcal{P}$. The rotations $\{E_{ij}\}_{i\neq 0}$ act trivially on $\mathbb{H}$ and ultimately we shall map the diffusion using $\pi : \mathbb{O}M \to \mathbb{H}$ $g \mapsto g^0$ and so we can ignore the trivial actions in which case we get the expression

$$\begin{aligned} dg_s &= \sum_{i=1}^{d} g_s^i E_i \circ dB_s^i \\ d\zeta_s &= g_s^0 ds. \end{aligned}$$

If we look at what the differential equation above does in a small time $\delta t$, we see that first the frames are transported along the direction $g_0$ keeping the frames parallel to this line. Once we arrive at $g_0\delta t$, then we apply a random change of bases and repeat the process again (this time the direction of $g^0$ may be different see Figure 3.1).

The following proposition will be proved in full generality in the next section, so we omit the proof.

**Proposition 3.1.2.** *The diffusion $(g_s, \zeta_s) \in \mathcal{P}$ given by the equations*

$$\begin{aligned} dg_s &= \sum_{i=1}^{d} g_s^i E_i \circ dB_s^i \\ d\zeta_s &= g_s^0 ds \end{aligned}$$

*where $B^i, i = 1, \cdots, d$ are independent Brownian motions on $\mathbb{R}$ generate a diffusion on $\mathbb{H}$ via the map $\pi((g, \zeta)) = g^0$ with the generator $\frac{1}{2}\Delta_{\mathbb{H}}$.*

The diffusion on $\mathbb{H}$ is then nothing but the projection of a process on the Lie group $\mathcal{P}$ with independent increments, viz. a Brownian motion with drift. Furthermore the same technique can also be applied to Lévy process on the Lie group $\mathcal{P}$. Using Corollary 2.3.6 we obtain a process on $\mathbb{H}$ with generator

$$\mathscr{L}f(e_0) = c\Delta_{\mathbb{H}}f(e_0) + \int_{\mathbb{H}}(f(\dot{\xi}) - f(e_0) - \sum_{i=1}^{d} y_i(\dot{\xi})\partial_{y_i}f(e_0))\, n(d\dot{\xi})$$

where $y_1, \ldots, y_d$ is a smooth co-ordinate system around $e_0$ and $n$ is measure on $\mathbb{H}$ that satisfies

$$\begin{aligned}
&n(\{e_0\}) = 0 \\
&n\left(\sum_{i=1}^{d} y_i^2\right) < \infty \\
&n(U^c) < \infty \quad \text{for any neighbourhood } U \text{ of } e_0.
\end{aligned}$$

## 3.2 Processes on Lorentz Manifolds

A Lorentz manifold $(M, g)$ is an orientable pseudo-Riemannian manifold $M$ with a metric which has signature $(1, -1, \cdots, -1)$.[4] In light of the previous section we wish to construct a process on $T^1M$, the positively oriented half of the unit sphere. Again we denote by $\mathbb{O}M$ the orthonormal frame bundle with its first element in $T^1M$ and $\pi : \mathbb{O}M \to T^1M$ maps each frame to its first element.

The same construction at the end of the preceding section does not work on Lorentz manifolds for two reasons; the frame bundle can no longer be identified with the Poincaré group and secondly, there may not be a global tangent space and the geodesics may be curved so $d\zeta_s = g^0 ds$ no longer makes sense. However, there is an analogous method for a similar construction on Lorentz manifolds and to do so we need some tools from geometry.

A connection may be used to connect tangent spaces and a natural choice is the Levi-Civita connection $\nabla$. The connection identifies curves in $\mathbb{O}M$ that are horizontal to $T^1M$ in the sense that if $\gamma \subset \mathbb{O}M$ is a curve,[5] then it is horizontal to $T^1M$ if for each $x \in \mathbb{R}^{1,d}$, $\nabla_{\pi(\gamma)}\gamma x = 0$ at each point of the curve. The space of vector fields that are horizontal is denoted by $H(\mathbb{O}M)$ and allow for $T(\mathbb{O}M)$ to be decomposed as

$$T(\mathbb{O}M) = H(\mathbb{O}M) \oplus V(\mathbb{O}M).$$

The elements of $V(\mathbb{O}M)$ are called the vertical vector fields. Intuitively thinking, the connection links up two tangent spaces in the sense that it allows us to identify vectors in one of them from the other. If a curve in $\mathbb{O}M$ is horizontal, it means that the frames transported parallel to the connection. In the case $\mathbb{R}^{1,d}$ we had that the connection was given by a straight line, and so we moved the frames staying parallel to the straight line $g^0$.

[4]The signature is the number of positive and negative eigenvalues of the metric. In this case, the metric generates 1 positive eigenvalue and $d$ negative eigenvalues.

[5]A curve on the orthonormal frame bundle is a smooth choice of frames at each point $p \in M$.

We can also “lift“ points in $T^1M$ to $\mathbb{O}M$. This is due to the fact that for each $X \in T^1_pM$ (the positively oriented half of unit sphere in $T_pM$) and $s_p \in \mathbb{O}M$ we have a unique horizontal vector $\tilde{X}$ on $\mathbb{O}M$ such that $\pi(\tilde{X}) = X$ (c.f. [Hsu02], [Mal97]).

Suppose $s_p \in \mathbb{O}M$ is an orthonormal frame on $T_pM$, then we can view $s_p : \mathbb{R}^{1,d} \to T_pM$, via $x \mapsto \sum_{i=0}^d x_i s_p^i$ where $s_p^0, \cdots, s_p^d$ are the frames. Using this, $SO^+(1,d)$ acts on $\mathbb{O}M$ in the following way

$$\mathbb{R}^{1,d} \xrightarrow{S0^+(1,d)} \mathbb{R}^{1,d} \xrightarrow{s_p} T_pM.$$

Indeed, the fibres of $\mathbb{O}M$ can be modelled on $SO^+(1,d)$. This action of $SO^+(1,d)$ on $\mathbb{O}M$ is simply transitive.[6]

As before, let $E_{ij} := e_i^* \otimes e_j - e_j^* \otimes e_i$ for $i, j = 0, \dots, d$ and $E_i := E_{0i}$. The vertical fibres of $\mathbb{O}M$ infinitesimally behave like $so^+(1,d)$ and so defining the vector fields $V_{ij}$ as the actions of $E_{ij}$, i.e.

$$V_{ij}f(u) := \frac{d}{dt} f(e^{tE_{ij}}u)\Big|_{t=0}$$

defines vertical vector fields which generate $V(\mathbb{O}M)$.[7]

Fix a starting point $u \in \mathbb{O}M$ and let $H_0$ be the horizontal lift of $\pi(u)$. In a small time period $\delta t$, we push the frames along $H_0$ for $\delta t$ time, then permute them with random noise using $V_{ij}$. As before, the vector fields $\{V_{ij} : 1 \leqslant i, j \leqslant d\}$ generate the rotations and so they act trivially on $T^1M$, hence we can restrict our attention to the vector fields $\{V_i\}_{i=1}^d$.

Formulating this, we have that the diffusions on $\mathbb{O}M$ are of the form

$$d\zeta_t = H_0(\zeta_s)ds + \sum_{i=1}^d V_i(\zeta_s) \circ dB_s^i \qquad \zeta_0 = u$$

where $B^1, \dots, B^d$ are independent Brownian motions on $\mathbb{R}$. The generator of the diffusion is then given by

$$H_0 + \frac{1}{2}\sum_{i=1}^d V_i^2.$$

Intuitively the operator $\sum_{i=1}^d V_i^2$ should give us some sort of a Laplace operator. Recall that a Laplace operator is given by $\Delta f = div\, grad\, f = trace(H(f))$ where $H(f)$ is the Hessian matrix, i.e. the matrix of second order derivations. The second derivations may be thought of as $V_iV_j$ as $E_iE_j$ are the derivatives in the Lie group around the identity, then the trace is precisely $\sum_{i=0}^d V_i^2$. The rest of the section will go towards proving this mathematically.

[6] I.e. free and transitive. The reason for this is that via a rotation, we can obtain one orthogonal frame from an other. In order for the frames to remain orthonormal we must apply $SO(1,d)$ and for the first element to remain in the positively oriented half, the action must be that of $SO^+(1,d)$.

[7] This is due to the fact that one can identify a one form between $\mathbb{O}M$ and $so^+(1,d)$ where the kernel contains the horizontal vector fields, c.f. [KN69]. Indeed this allows $V_{ij}$ to generate all the vertical vector fields on $\mathbb{O}M$.

If we let $g_{ij} = \langle(\partial/\partial x_i), (\partial/\partial x_j)\rangle$ and $g^{ij}$ be the inverse of $g_{ij}$ then the Laplace-Beltrami operator can be expressed locally as

$$\Delta_M = \frac{1}{2}g^{ij}\left(\frac{\partial^2}{\partial x_i \partial x_j} - \Gamma^k_{ij}\frac{\partial}{\partial x_k}\right)$$

where $\Gamma^k_{ij}$ are the Christoffel symbols given by

$$\Gamma^k_{ij}\partial_k = \nabla_{d\partial_i}\partial_j$$

where $\partial_i = (\partial/\partial x_i)$.

The following theorem then closes off this section nicely.

**Theorem 3.2.1.** *The operator $\mathcal{V} := \sum_{i=1}^{d} V_i^2$ induces on $\mathscr{C}^2(T^1M)$ the Laplacian $\Delta_v$ in the sense that for any $f \in \mathscr{C}^2(T^1M)$*

$$(\Delta_M f) \circ \pi = \mathcal{V}(f \circ \pi)$$

*Proof.* Let $e_0, \ldots, e_d$ be the canonical basis for $\mathbb{R}^{1,d}$. The exponential matrix $e^{tE_i}$ is given by[8]

$$\begin{pmatrix} \cosh t & 0 & \cdots & \sinh t & 0 & \cdots \\ 0 & 1 & 0 & \cdots & 0 & \cdots \\ \vdots & & \ddots & & & \\ \sinh t & & & \cosh t & & \\ 0 & & & & 1 & \\ \vdots & & & & & \ddots \end{pmatrix}$$

and so we have that the only terms that change are

$$\begin{aligned} e^{tE_i}e_0 &= e_0 \cosh t + e_i \sinh t \\ e^{tE_i}e_i &= e_i \cosh t + e_0 \sinh t \end{aligned}$$

and so

$$\begin{aligned} \frac{d}{dt}e^{tE_i}e_0\Big|_{t=0} &= e_i \\ \frac{d}{dt}e^{tE_i}e_i\Big|_{t=0} &= e_0 \end{aligned}$$

with $(d/dt)e^{tE_i}e_j = 0$ for $j \neq i, 0$.

[8]This and a few other computations have been left out as they can be confirmed with the aid of a computer thus add no real value to the proof.

Choose the coordinate system $(x^j, e^j)$, $e^j = (e_0^j, \ldots, e_d^j)$ in $\mathbb{O}M$ such that $e_i = e_i^k(\partial/\partial x^k)$ then for any $f \in \mathscr{C}^2(\mathbb{O}M)$ using the chain rule we have

$$\begin{aligned} V_i f(u_x) &= \frac{d}{dt} f(e^{tE_i} u_x)\Big|_{t=0} = \frac{de^{tE_i} e_0^k}{dt}\Big|_{t=0} \frac{\partial}{\partial e_0^k} f(u_x) + \frac{de^{tE_i} e_i^k}{dt}\Big|_{t=0} \frac{\partial}{\partial e_i^k} f(u_x) \\ &= e_0^k \frac{\partial}{\partial e_i^k} f(u_x) + e_i^k \frac{\partial}{\partial e_0^k} f(u_x). \end{aligned}$$

Hence the action of $V_i$ is given by

$$V_i = e_0^k \frac{\partial}{\partial e_i^k} + e_i^k \frac{\partial}{\partial e_0^k}.$$

Using the fact $\sum_{i=1}^d e_i^k e_i^l + g^{kl} = e_0^k e_0^l$ one can directly compute

$$\mathcal{V} f \circ \pi = \left( (e_0^k e_0^l - g^{kl}) \frac{\partial^2}{\partial e_0^k e_0^l} + de^k \frac{\partial}{\partial e_0^k} \right) f \circ \pi.$$

□

## 3.3 Asymptotic Behaviour in Minkowski Spacetime

If we fix the point $p = (1,0,0,0) \in \mathbb{H}$ and take $x = (x_0, x^*) \in \mathbb{H}$, then $x$ may be expressed in polar co-ordinates $(\rho, \sigma) \in \mathbb{R}_+ \times \mathbb{S}^{d-1}$ by $\rho = \cosh^{-1}(x_0)$ and $\sigma = x^*/\sqrt{x_0^2 - 1}$. The asymptotic properties of $\dot{\xi}_s$ is indeed different than a Brownian motion on a Euclidian space. The described set up was established by Dudley in [Dud65] who proved that the radial part of the process $\rho_s$ is transient. In [Bai08b] and [BR08] describe more properties of this process. The latter shall provide a reference for the remainder of section where we will prove that the angular part $\sigma_s$ of the process $\dot{\xi}_s$ converges to some random angle $\sigma_\infty$, which in turn implies that $\xi_s$ asymptotically approaches a hyperplane at a random height, see Figure 3.3.

**Theorem 3.3.1.** *Writing out $\dot{\xi}$ in polar co-ordinates $(\rho_t, \sigma_t)$, we have that*

$$\begin{aligned} \rho_t &= \rho_0 + \beta_t + \frac{d-1}{2} \int_0^t \coth \rho_s ds \\ \sigma_t &= \Sigma \left( \int_0^t \frac{ds}{\sinh^2 \rho_s} \right) \end{aligned} \tag{3.3.1}$$

*where $\beta$ and $\Sigma$ are independent Brownian motions on the reals and $\mathbb{S}^{d-1}$ respectively.*

*Proof.* Notice that $q = d\rho^2 + \sinh^2 \rho \, d\theta^2$ where $d\theta^2$ is the Reimannian metric on $\mathbb{S}^{d-1}$. A direct computation shows that the Laplace-Beltrami operator on $\mathbb{H}$ can be expressed as;

$$\Delta_{\mathbb{H}} = \left( \frac{\partial}{\partial \rho} \right)^2 + (d-1) \coth \rho \frac{\partial}{\partial \rho} + \frac{\Delta_{\mathbb{S}^{d-1}}}{\sinh^2 \rho} \tag{3.3.2}$$

where $\Delta_{\mathbb{S}^{d-1}}$ is the Laplace-Beltrami operator on $\mathbb{S}^{d-1}$.

To obtain the first equation use the map $r(x) = \rho$ and the fact that $\dot{\xi}$ is a $1/2\Delta_{\mathbb{H}}$-diffusion to get that

$$\beta_t = \rho_t - \rho_0 - \frac{1}{2}\int_0^t \left(\frac{\partial}{\partial \rho}\right)^2 + (d-1)\coth\rho\frac{\partial}{\partial\rho} r(\dot{\xi}_s)ds$$

is a continuous local martingale. The action on $f$ can be shown via direct computation to be $(d-1)/2\coth$ and so the only remaining task is to show that $\beta$ is a Brownian motion. Notice that we have;

$$\frac{1}{2}\Delta_{\mathbb{H}}r^2 - r\Delta_{\mathbb{H}}r = \frac{1}{2}(2 + 2(d-1)\rho\coth\rho - 2(d-1)\rho\coth\rho) = 1$$

and so the quadratic variation of $\beta$ is

$$[\beta]_t = \int_0^t \frac{1}{2}(\Delta_{\mathbb{H}}r^2 - r\Delta r)(\dot{\xi}_s)ds = t$$

and hence $\beta$ is Brownian motion by Lévy's characterisation.

Similarly by taking $f(x) = \sigma$ gives that

$$M_t^f := \sigma_t - \sigma_0 - \frac{1}{2}\int_0^t \frac{\Delta_{\mathbb{S}^{d-1}}}{\sinh^2\rho} f(\dot{\xi}_s)_s ds$$

is an $\mathbb{S}^{d-1}$-valued continuous local martingale. Let $\tau_t$ be the right inverse of $t \mapsto \int_0^t \frac{ds}{\sinh^2\rho_s}$, then $\Sigma_t = M_{\tau_t}^f$ is a time change Brownian motion by Dubins-Schwarz theorem.

For the independence we have that

$$\begin{aligned}
\Delta_{\mathbb{H}}(fr) &= (d-1)\sigma\coth\rho + \frac{\rho\Delta_{\mathbb{S}^{d-1}}f}{\sinh^2\rho} \\
f\Delta_{\mathbb{H}}r &= (d-1)\sigma\coth\rho_0 \\
r\Delta_{\mathbb{H}}f &= \frac{\rho\Delta_{\mathbb{S}^{d-1}}f}{\sinh^2\rho}
\end{aligned}$$

and so

$$[\beta, \Sigma]_t = \Delta_{\mathbb{H}}(fr) - r\Delta_{\mathbb{H}}(f) - f\Delta_{\mathbb{H}}(r) = 0.$$

□

Using this theorem we may now prove the intended result.

**Theorem 3.3.2.** *Suppose that $\dot{\xi} = (\dot{\xi}_t : t \geqslant 0)$ is a Brownian motion on $\mathbb{H}$ with polar co-ordinates $(\rho_t, \sigma_t)$ and $\xi = (\xi_t : t \geqslant 0)$ given by $\xi_t = \xi_0 + \int_0^t \dot{\xi}_{s-}\, ds$, then*

(i) *$\rho_t$ is transient*

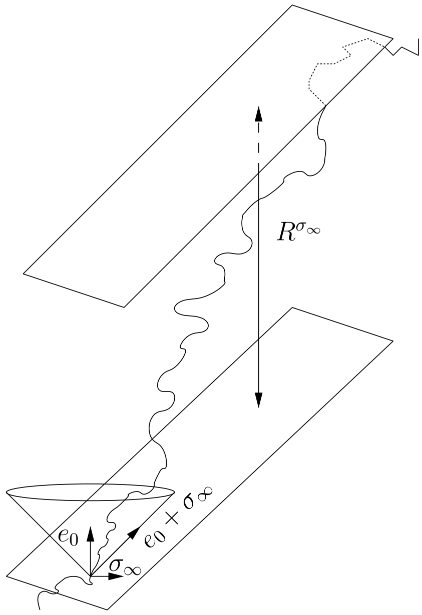


Figure 3.2: Asymptotic behaviour of Brownian motion in $\mathbb{R}^{1,d}$

(ii) $\sigma_\infty := \lim_{t\to\infty} \sigma_t$ *exists almost surely and moreover if the process is started from* $e_0$ *then* $\sigma_\infty$ *is uniformly distributed on* $\mathbb{S}^{d-1}$

(iii) *The random height* $R^{\sigma_\infty} := \lim_{t\to} q(\xi_t, e_0 + \sigma_\infty)$ *exists and so the process* $\xi$ *approaches asymptotically the random hyperplane parallel to* $e_0 + \sigma_\infty$ *at hight* $R^{\sigma_\infty}$

*Proof.* (i) Due to the fact that coth $\geqslant 1$ and the monotonicity of the integral,

$$\rho_t \geqslant \rho_0 + \beta_t + \frac{d-1}{2}\int_0^t ds = \rho_0 + \beta_t + \frac{d-1}{2}t$$

By the strong law of large numbers and Donsker's invariance principle the process $\beta_t + \frac{d-1}{2}t \to \infty$ as $t \to \infty$ and hence the result follows.[9]

(ii) The convergence is an immediate consequence of the preceding theorem, part (i) and the law of the iterated logarithm which ensures that $\beta_t$ fluctuates as $\sqrt{2t \log\log t}$ for $t$ large enough.[10] The distribution of $\sigma_\infty$ started from $e_0$ is uniform because the law of the Brownian motion is invariant under the rotations and as the rotations fix $e_0$ the law of $\sigma_\infty$ must be invariant under rotations. The uniform distribution is the unique distribution on $\mathbb{S}^{d-1}$ with this property. □

[9]The alternative is to let $T_a := \inf\{t > 0 : \beta_t + ct = a\}$ where $c = (d-1)/2$. Now for $a < 0 < b$ by Girsanov's theorem we have $\mathbb{P}(T_b < T_a) = \frac{1-e^{2ac)}}{1-e^{-2(b-a)c}}$. By letting $a \downarrow -\infty$ we have that $\mathbb{P}(T_b < \infty) = 1$ and by $b \uparrow \infty$ we have $\mathbb{P}(T_a < \infty) < 1$. Now we can apply the strong Markov property to conclude the result.

[10]The law of the iterated logarithm states that $\overline{\lim}\frac{\beta_s}{\sqrt{2s\log\log s}} = 1$ and consequently via symmetry $\underline{\lim}\frac{\beta_s}{\sqrt{2s\log\log s}} = -1$ a.s., c.f. [KS91]

For the moment we cannot prove the last part of the statement. In the next section we shall become acquainted with the tools necessary to solve this problem.

## 3.4 h-Transformations

We will prove that the limit $\lim_{t\to\infty} q(\xi_t, e_0 + \sigma_\infty)$ exists by conditioning on the process hitting a particular value of $\sigma_\infty$. Unlike previously, in this section we start the diffusions at some point $(\xi, \dot{\xi}) \in \mathbb{R}^{1,d} \times \mathbb{H}$ and denote the law by $\mathbb{P}_{(\xi,\dot{\xi})}$ or where appropriate we will drop one of the coordinates and simply write $\mathbb{P}_\xi = \mathbb{P}_{(\xi,\dot{\xi})}(\cdot \times \mathbb{H})$ or $\mathbb{P}_{\dot{\xi}} = \mathbb{P}_{(\xi,\dot{\xi})}(\mathbb{R}^{1,d} \times \cdot)$.

Suppose that a process has generator $L$ and that $h$ is a positive, bounded $L$-harmonic function, that is $Lh = 0$.[11] The $h$-transform of a process is an other process that has generator $L^h f := (L(hf))/h$.

Informally, we will take as our harmonic function $h^\sigma(\dot{\xi}) = \mathbb{P}_{\dot{\xi}}(\sigma_\infty = \sigma)$. More formally, this will be the density of $\sigma_\infty$ with respect to the uniform measure $d\alpha$ on $\mathbb{S}^{d-1}$.

**Proposition 3.4.1.** *For any $(\xi, \dot{\xi}) \in \mathbb{R}^{1,d} \times \mathbb{H}$, the distribution of $\sigma_\infty$ under $\mathbb{P}_{(\xi,\dot{\xi})}$ is continuous with respect to the uniform measure $d\alpha$ on $\mathbb{S}^{d-1}$ and thus admits a density $h^\sigma$.*

*Proof.* This follows easily from Theorem 3.3.1 as Brownian motion is continuous with respect to the uniform measure on $\mathbb{S}^{d-1}$ regardless of the starting position. □

Now we assume that $\sigma \in \mathbb{S}^{d-1}$ is fixed. To simplify matters, sometimes it is more convenient to work in the Poincaré half plane model $\mathbb{R}_* \times \mathbb{R}^{d-1}$. The Riemannian metric at a point $(y, x) \in \mathbb{R}_* \times \mathbb{R}^{d-1}$ is given by

$$X, Y \in \mathbb{R}^d \qquad \langle X, Y\rangle_{(y,x)} = \frac{\langle X, Y\rangle_E}{y^2}$$

where $\langle \cdot, \cdot\rangle_E$ is the Euclidian scalar product. This is a model for the hyperbolic space $\mathbb{H}$ as the following map is an isometry;

$$\mathbb{H} \ni (\dot{\xi}^0, \cdots, \dot{\xi}^d) \overset{\psi^{-1}}{\longmapsto} \left(\frac{1}{\dot{\xi}^0 - \dot{\xi}^1}, \frac{\dot{\xi}^2}{\dot{\xi}^0 - \dot{\xi}^1}, \cdots, \frac{\dot{\xi}^d}{\dot{\xi}^0 - \dot{\xi}^1}\right) \in \mathbb{R}_* \times \mathbb{R}^{d-1}$$
$$\mathbb{R}_* \times \mathbb{R}^{d-1} \ni (y, x) \overset{\psi}{\longmapsto} \left(\frac{||x||_E^2 + y^2 + 1}{2y}, \frac{||x||_E^2 + y^2 - 1}{2y}, \frac{x_1}{y}, \cdots, \frac{x_{d-1}}{y}\right) \in \mathbb{H}$$

where $||\cdot||_E$ is the Euclidian norm.

The Laplace-Beltrami operator on $\mathbb{R}_* \times \mathbb{R}^{d-1}$ is given by

$$\Delta_{\mathbb{H}} = y^2(\partial_{x_1}^2 + \cdots + \partial_{x_{d-1}}^2 + \partial_y^2) - (d-2)y\partial_y.$$

[11] No generality is lost in assuming that the harmonic functions are positive. Indeed for any bounded harmonic function $h$, $h - \inf h$ is also harmonic.

As the diffusion on $\mathbb{H}$ has generator $(1/2)\Delta_{\mathbb{H}}$, then by denoting $\partial^{\dot{\xi}} f(\xi, \dot{\xi})$ as the derivative of $\xi$ along $\dot{\xi}$, we have

$$Lf := \frac{\Delta_{\mathbb{H}} f}{2} + \partial^{\dot{\xi}} f.$$

In [Pin95][Theorem 9.5.2] Pinsky proves that for any measurable set $U \subset \mathbb{S}^{d-1}$, $(\xi, \dot{\xi}) \mapsto \mathbb{P}_{(\xi,\dot{\xi})}(\sigma_\infty \in U)$ defines a harmonic map and in particular this implies that $h^\sigma$ is harmonic. Let $L^{h^\sigma}$ denote the h-transform, i.e. $L^{h^\sigma} f = L(hf)/h$, and $p(t, x, dy)$ be the transition kernels of $(\xi_s, \dot{\xi}_s)$. Define

$$p^{h^\sigma}(t, x, dy) = \frac{h^\sigma(y)}{h^\sigma(x)} p(t, x, dy)$$

The process given by the transition kernels has generator $L^{h^\sigma}$ and[12] thus solves the $L^{h^\sigma}$ martingale problem

$$M^f = f(\xi_t, \dot{\xi}_t) - \int_0^t L^{h^\sigma} f(\xi_s, \dot{\xi}_s) ds$$

and hence must be strongly Markovian. The continuity of the process is apparent from the kernels.

*Proof of Theorem 3.3.2 (iii).* We can find more information about $h^\sigma$ by looking at the Poincaré ball model of the hyperbolic space. Let $\mathbb{B}^d \subset \mathbb{R}^d$ be the open unit ball. The hyperbolic space can be thought of as living in $\mathbb{B}^d$ by using the following map to induce a metric on it (c.f. [BH99])

$$\begin{aligned} \mathbb{H} \ni (\dot{\xi}^0, \cdots, \dot{\xi}^d) &\xmapsto{\phi^{-1}} \left( \frac{\dot{\xi}^1}{1 + \dot{\xi}^0}, \cdots, \frac{\dot{\xi}^d}{1 + \dot{\xi}^0} \right) \in \mathbb{B}^d \\ \mathbb{B}^d \ni x &\xmapsto{\phi} \left( \frac{1 + ||x||_E^2}{1 - ||x||_E^2}, \frac{2x}{1 - ||x||_E^2} \right) \in \mathbb{H}. \end{aligned}$$

Notice that this map preserves the conformal boundary, i.e. $\phi(\sigma) = \sigma$ for each $\sigma \in \mathbb{S}^d$. The corresponding map between the half-plane $\mathbb{R}_* \times \mathbb{R}^{d-1}$ and $\mathbb{B}^d$ is then given by;

$$\begin{aligned} \mathbb{R}_* \times \mathbb{R}^{d-1} \ni (y, x) &\xmapsto{\Phi^{-1}} \left( \frac{||x||_E^2 + y^2 - 1}{||x||_E^2 + (1 + y)^2}, \frac{2x}{||x||_E^2 + (1 + y)^2} \right) \in \mathscr{B}^d \\ \mathscr{B}^d \ni x &\xmapsto{\Phi} \left( \frac{||x||_E^2 - 1}{1 - 2x_1 + ||x||_E^2}, \frac{2x_d}{1 - 2x_1 + ||x||_E^2}, \dots, \frac{2x_2}{1 - 2x_1 + ||x||_E^2} \right) \in \mathbb{R}_* \times \mathbb{R}^{d-1}. \end{aligned}$$

The Poisson kernel[13] $K(x, \eta)$ is well known in the ball and is given by (c.f. [Doo84])

$$K(x, \eta) = \frac{1 - ||x||_E^2}{||\eta - x||_E^d}.$$

[12] consult [Pin95][Theorem 4.1.1]

[13] Probabilistically, this is the density of a Brownian motion starting at $x \in \mathbb{B}^d$ and hitting $\eta \in \mathbb{S}^d$.

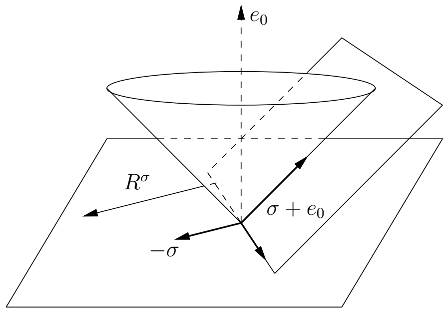


Figure 3.3: New random co-ordinates on $\mathbb{R}^{1,d}$ given by $\mathrm{g}'$

Suppose now we tilt the axis by defining a new basis $\{\epsilon_0, \cdots, \epsilon_d\} \subset \mathbb{R}^{1,d}$ by letting $\epsilon_0 = e_0$, $\epsilon_1 = \sigma$ and $\epsilon_2, \cdots, \epsilon_d$ be orthonormal. To avoid confusion, we let $\dot{\xi} = (\dot{\xi}^0, \ldots, \dot{\xi}^d)$ be in the new co-ordinate system. We can also specify, as above, half-space coordinates by letting $\mathrm{g} = (\epsilon_0, \epsilon_1, \cdots, \epsilon_d)$ and $(y, x) = \psi^{-1}(\mathrm{g}^{-1}\xi)$. The Laplacian on $\mathbb{R}_* \times \mathbb{R}^{d-1}$ with the new co-ordinates has the same expression has above.

Notice that the point $\sigma$ in these co-ordinates corresponds to $(1, 0, \ldots, 0) \in \mathbb{B}^d$, which corresponds to $\infty \in \partial(\mathbb{R}_* \times \mathbb{R}^{d-1})$, where $\partial A$ is the conformal boundary of $A$. The hitting density $h^{\epsilon_1}(y, x)$ then is invariant under translations in $x \in \mathbb{R}^{d-1}$ so it suffices to compute $h^{\epsilon_1}(y, 0)$. Using the kernel on $\mathbb{B}^d$, we obtain that $h^{\epsilon_1}$ is proportional to a polynomial of degree $d-1$, and using the fact that $\Delta_{\mathbb{H}} h^{\epsilon_1} = 0$ we obtain that $h^{\epsilon_1}$ is proportional to $y^{d-1}$. Thus

$$L^{h^{\epsilon_1}} f = \frac{y^2}{2}(\partial_x^2 f + \partial_y^2 f) + \frac{d}{2} y \partial_y f + \partial^{\dot{\xi}} f$$

The $y$ terms are nothing but the description of an Itô diffusion in $\mathbb{R}$ and so this has the expression

$$dy_s = y_s dB_s + \frac{d}{2} y_s ds$$

where $B = (B_t : t \geqslant 0)$ is a Brownian motion in $\mathbb{R}$. This is just a geometric Brownian motion.

If we change the co-ordinates once more using $\mathrm{g}' := (-\epsilon_1, \epsilon_0 + \epsilon_1, \epsilon_2, \ldots, \epsilon_d)$, we see that now that it is sufficient to find the bound of the processes in these co-ordinates (see Fig3.4). Let $(\xi'^0, \ldots, \xi'^d)$ be $\xi \in \mathbb{H}$ expressed in the co-ordinates given by $\mathrm{g}'$, then

$$d\xi'^0_s = \frac{ds}{y_s} = \frac{ds}{y_0 e^{B_s + \frac{d-1}{2}s}}.$$

On the other hand, observe that $\xi'^0 = q(\xi, \sigma_\infty + e_0)$ and so recalling that $R^{\sigma_\infty} = \lim_{s\to\infty} q(\xi_s, \sigma_\infty + e_0)$, we have

$$R^{\sigma_\infty} = \xi'_0 + \frac{1}{y_0}\int_0^\infty e^{-B_s - \frac{d-1}{2}s} ds.$$

It remains to see that the integral is finite. This is a consequence of the fact that as noted before $B_s + Cs \to \infty$ as $s \uparrow \infty$, where $C$ is a constant, and the law of the iterated logarithm.

$\square$

# POISSON BOUNDARY

In the last section we encountered $h^\sigma$ which is a bounded harmonic function that is non-constant. Indeed this contrasts starkly with the Euclidian case where Liouville's theorem states that any bounded harmonic function must be constant. To keep notation to a minimum we denote by $(\xi_s, \dot{\xi}_s)$ the diffusion described as before. The invariant $\sigma$-algebra $Inv((\xi_s, \dot{\xi}_s))$ given by events of the form $(\xi_s, \dot{\xi}_s) \in A$ if and only if $(\xi_{s+t}, \dot{\xi}_{s+t}) \in A$ for each $t \geqslant 0$.

**Theorem 4.0.2** (Correspondence Between Probability and Analysis). *If $h$ is a bounded $L$-harmonic function then there exists an $X \in Inv((\xi_s, \dot{\xi}_s))$ such that*

$$h(\xi, \dot{\xi}) = \mathbb{E}_{(\xi,\dot{\xi})}[X].$$

*Conversely, any function in the above form for some $X \in Inv((\xi_s, \dot{\xi}_s))$ is $L$-harmonic.*

*Proof.* Suppose that $h$ is bounded and $L$-harmonic, then by Itô's lemma $h((\xi_s, \dot{\xi}_s))$ is a bounded Martingale and thus by the Martingale convergence theorem, converges to a random variable $X$ a.s. It is clear that $X \in Inv((\xi_s, \dot{\xi}_s))$ and moreover by the optional stopping theorem $\mathbb{E}_{\mathbf{e}}[X] = h(\mathbf{e})$.

Conversely take $X \in Inv((\xi_s, \dot{\xi}_s))$ and denote by $\theta_s$ the shift operator, then

$$P_t \mathbb{E}_{(\xi,\dot{\xi})}[X] = \mathbb{E}_{(\xi,\dot{\xi})}[\mathbb{E}_{(\xi_s,\dot{\xi}_s)}[X]] = \mathbb{E}_{(\xi,\dot{\xi})}[X \circ \theta_s] = \mathbb{E}_{(\xi,\dot{\xi})}[X]$$

which shows that $(\xi, \dot{\xi}) \mapsto \mathbb{E}_{(\xi,\dot{\xi})}[X]$ is harmonic. □

Thus the invariant $\sigma$-algebra determines the behaviour of the harmonic functions. One has a-priori that $Inv((\xi_s, \dot{\xi}_s)) \subset Tail((\xi_s, \dot{\xi}_s))$ and so in the case of some manifolds, the tail $\sigma$-algebra fails to be trivial.

It is readily seen now that it is enough for the angular part of a process to converge in order to obtain a non-constant bounded harmonic function.[1]

The Poisson boundary is the set of bounded harmonic functions and as before, no generality is lost in assuming that these functions are positive. Most of the discussion so far has been towards the behaviour of the process in $\mathbb{R}^{1,d} \times \mathbb{H}$. Indeed this relates to the Poisson boundary via the following;

(i) The $\sigma$-algebras $Inv(\xi_s, \dot{\xi}_s)$, $Tail(\xi_s, \dot{\xi}_s)$ and $\sigma(\sigma_\infty, R^{\sigma_\infty})$ co-inside $\mathbb{P}_{(\xi,\dot{\xi})}$-a.s.

[1] Note that the process is invariant under isometries, and hence if the angular part converges then it will be invariant under rotations (given that the rotations leave the origin invariant) so the invariant algebra will be non-trivial.

(ii) The joint law of $(\sigma_\infty, R^{\sigma_\infty})$ is continuous with respect to $d\alpha\, dl$, the product measure of the uniform measure on $\mathbb{S}^{d-1}$ and the Lebesgue measure on $\mathbb{R}$, and admits a density $h^\sigma(\xi,\dot{\xi})h_l^\sigma(\xi,\dot{\xi})$

(iii) For any starting point $(\xi,\dot{\xi}) \in \mathbb{R}^{1,d} \times \mathbb{H}$, the positive function $h^\sigma h_l^\sigma$ is a minimal $L$-harmonic function, in the sense that if $u \leqslant h^\sigma h_l^\sigma$ is harmonic, then $u = Ch^\sigma h_l^\sigma$ for some constant $C$

(iv) Consequently a positive bounded harmonic function is of the form $\int\int Fh^\sigma h_l^\sigma d\alpha dl$, where $F : \mathbb{S}^{d-1} \times \mathbb{R} \to \mathbb{R}$ is Borel measurable, and conversely any function of that form for some Borel measurable $F$ is a positive bounded harmonic function.

There are two methods of proving these statements found in [Bai08b] and [BR08].

In [Bai08b] Bailleul uses the coupling method to prove these statements. The idea of coupling is to look at a pair of stopping times $(S,T)$ such for processes $(\xi_s, \dot{\xi}_s)$, $(\underline{\xi}_s, \dot{\underline{\xi}}_s)$ with starting distributions $\mu$ and $\nu$ respectively, couple at $(S,T)$ i.e. $\mathbb{P}((\xi_S, \dot{\xi}_S) = (\underline{\xi}_T, \dot{\underline{\xi}}_T)) = 1$. Coupling then allows us to express;

$$\mathbb{P}(T=\infty) + \mathbb{P}(S=\infty) = \sup\{\langle \mu-\nu, h\rangle : h \geqslant 0 \text{ harmonic function }, ||h|| < 1\}$$

where the norm is the uniform Banach norm (see [CG95]).

Bailleul shows that bounded $L^{h^\sigma h_l^\sigma}$-harmonic functions are constant by showing that the processes with generator $L^{h^\sigma h_l^\sigma}$ couple in a finite time. Through Theorem 4.0.2 this shows that the tail $\sigma$-algebra of the $L^{h^\sigma h_l^\sigma}$-diffusion is trivial. It is well known (see [Doo84] for instance) that the $h$-transform corresponds to the process conditions on hitting a conformal boundary point, hence this shows that indeed the invariant $\sigma$-algebra is given by $\sigma(\sigma_\infty, R^{\sigma_\infty})$.

An alternative approach is given in [BR08], where Bailleul and Raugi look at the random walk on the Poincaré group $\mathcal{P}$. Naturally, $\mathcal{P}$ is just $\mathbb{R}^{1,d} \times SO^+(1,d)$ and the map $\mathcal{P} \ni (\xi, \mathrm{g}) \mapsto (\xi, \mathrm{g}^0) \in \mathbb{R}^{1,d} \times \mathbb{H}$ helps us project results.

The Iwasawa decomposition allows for a decomposition of the Lie algebra $so^+(1,d)$ into $\mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ where the and the associated groups $K$, $A$ and $N$. This is similar to the Cartan decomposition in the sense that $K \times A \times N \to SO^+(1,d)$, $(k,a,n) \mapsto kan$ is a diffeomorphism. The algebras are given by

$$\mathfrak{k} = so(d)$$

$$\mathfrak{a} \text{ is generated by } E_1 = e_0^* \otimes e_1 - e_1^* \otimes e_0$$

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & 0 & x^T & 0 & \cdots \\ 0 & 0 & x^T & 0 & \cdots \\ x & -x & 0 & 0 & \cdots \\ 0 & 0 & 0 & \ddots & \end{pmatrix} : x \in \mathbb{R}^{d-1} \right\}.$$

This allows one to decompose the Lie group $\mathcal{P}$ into $D^- \times D^+ \times SO(d)$ where $D^- = \mathbb{R}(e_0+e_1) \times N$ and $D^+ = (\mathbb{R}(e_0 - e_1) \oplus \mathrm{span}\{e_2, \dots, e_d\}) \times A$. Again the map $D^- \times D^+ \times$

$SO(d) \ni (d^-, d^+, k) \mapsto d^- d^+ k \in \mathcal{P}$ is a diffeomorphism. A random walk $\mathbf{e}_n = (\xi_n, \mathrm{g}_n) \in \mathcal{P}$ can now be split into $\mathbf{e}_n = d_n^- d_n^+ k_n$ where $d_n^- \in D^-$, $d_n^+ \in D^+$ and $k_n \in SO(d)$.

Bailleul and Raugi then prove that the limit $\lim_{s\to\infty} d_s^-$ exists $\mathbb{P}_{\mathbf{e}}$-a.s. and that any $D^-$-left invariant bounded harmonic function is constant. Indeed this convergence is much similar to that of the $h$-transform method, and allows for a similar expression for the bounded harmonic functions.

The Poisson boundary of general Lorentz manifolds is still unknown. Though it is known that on complete manifolds with non-negative Ricci curvature, Liouville's theorem holds, i.e. every bounded harmonic function is constant (c.f. [Hsu88]). Perhaps one could, on some spaces, associate the $L^{h^\sigma h_l^\sigma}$-diffusion with a $\Delta_M$-diffusion on a complete manifold $M$ with non-negative Ricci curvature and obtain results similar to that of Bailleul and Raugi.

We leave the reader with the following epilogue;

> "The most beautiful thing we can experience is the mysterious. It is the source of all true art and all science. He to whom this emotion is a stranger, who can no longer pause to wonder and stand rapt in awe, is as good as dead: his eyes are closed."
>
> -Albert Einstein

# Bibliography


[AK93] David Applebaum and Hiroshi Kunita. Lévy flows on manifolds and Lévy processes on Lie groups. *J. Math. Kyoto Univ*, 33(1103):23, 1993.

[App04] David Applebaum. *Lévy Processes and Stochastic Calculus.* Cambridge University Press, 2nd edition, 2004.

[Bai08a] I. Bailleul. A pathwise approach to relativistic diffusions. 2008.

[Bai08b] I. Bailleul. Poisson boundary of a relativistic diffusion. *Probability Theory and Related Fields*, 141(1):283–329, 2008.

[Ber96] Jean Bertoin. *Lévy Processes.* Cambridge University Press, 1996.

[BH99] M.R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature.* Springer Verlag, 1999.

[Bis84] J.M. Bismut. The Atiyah–Singer theorems: A probabilistic approach. I. The index theorem. *Journal of Functional Analysis*, 57(1):56–99, 1984.

[BR08] I. Bailleul and A. Raugi. Where does randomness lead in spacetime? *ESAIM: Probability and Statistics*, 14:16–52, 2008.

[CG95] M. Cranston and A. Greven. Coupling and harmonic functions in the case of continuous time Markov processes. *Stochastic processes and their applications*, 60(2):261–286, 1995.

[Dav75] B. Davis. Picard's theorem and Brownian motion. *Transactions of the American Mathematical Society*, 213:353–362, 1975.

[Doo84] J.L. Doob. *Classical potential theory and its probabilistic counterpart.* Springer, 1984.

[Dud65] Richard M. Dudley. Lorentz invariant markov processes in relativistic phase space. *Arkiv For Matematik*, 6(14), 1965.

[FLJ07] Jacques Franchi and Yves Le Jan. Relativistic Diffusions and Schwarzschild Geometry. *Communications on Pure and Applied Mathematics*, LX:187–251, 2007.

[Hsu88] P. Hsu. Brownian motion and Riemannian geometry. *Contemp. Math*, 73:95–104, 1988.

[Hsu02] E.P. Hsu. *Stochastic analysis on manifolds*. Amer Mathematical Society, 2002.

[Hun56] Gilbert Agnew Hunt. Semi-groups of measures on Lie groups. *Transactions of the American Mathematical Society*, 81(2):264–293, 1956.

[KN69] S. Kobayashi and K. Nomizu. Foundations of Differential Geometry, Vol. 1 & 2. *Interscience Tracts in Pure and Applied Mathematics, Interscience Publ., New-York*, 1969.

[KS91] I. Karatzas and S.E. Shreve. *Brownian motion and stochastic calculus*. Springer, 1991.

[Lia04] Ming Liao. *Lévy Processes in Lie Groups*. Cambridge University Press, 2004.

[Mal97] P. Malliavin. *Stochastic analysis*. Springer Verlag, 1997.

[Pin95] R.G. Pinsky. *Positive harmonic functions and diffusion*. Cambridge Univ Pr, 1995.

[Ram74] S. Ramaswami. Semigroups of measures on Lie groups. *J. Indian Math. Soc*, 38:175–189, 1974.

[RY99] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*. Springer Verlag, 1999.

[Sat99] Ken-iti Sato. *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, 1999.